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A characterization of the condition pseudospectrum on Banach space

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Abstract. We intend, through this paper, to extend the concept of condition pseudospectrum to the case of bounded linear operators on Banach spaces and prove several relations to the usual spectrum. We will begin by defining it and then we focus on the characterization, the stability and some properties of these spectra.

1. Introduction

Let $\mathcal{L}(X)$ denote the space of all bounded linear operators acting in a Banach space X and let $\sigma(T)$ (resp. $\rho(T)$) denote the spectrum (resp. the resolvent) of an operator $T \in \mathcal{L}(X)$. We denote by T' (resp. $T = I$) the adjoint operator (resp. the identity operator). The pseudospectrum of $T \in \mathcal{L}(X)$ is usually defined as:

$$\sigma_\varepsilon(T) := \sigma(T) \bigcup \left\{ \lambda \in \mathbb{C} : \ \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

and coincides with the set

$$\bigcup \left\{ \sigma(T + D) : D \in \mathcal{L}(X) \ \text{ and } \ \|D\| < \varepsilon \right\} \quad \text{(see [5])}$$

where, $\varepsilon > 0$ and $\|(\lambda - T)^{-1}\|$ is assumed to be infinite, if $\lambda - T$ is not invertible. Note that because of this convention, $\sigma(T) \subseteq \sigma_\varepsilon(T)$ for every $\varepsilon > 0$. For further details about pseudospectrum, we can refer to [1–5]. Firstly, L. N. Trefethen [13, 14] was the pioneer who introduced this concept while developing this idea for matrices and operators and who applied it to several highly interesting problems. Then, other several mathematicians continued their studies in this field. We can cite J. M. Varah [15] and E. B. Davies [5]. In this research we will consider one more such extension in terms of the condition number. Let X be a complex Banach space. The condition number of an invertible operators is defined as $\|T\|\|T^{-1}\|$, it is convenient to make a convention that $\|T\|\|T^{-1}\| = \infty$, if a is not invertible.

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From here, the following definition suggests itself. The definition of condition pseudospectra of a linear operator T in infinite dimensional Banach spaces is given for every $0 < \varepsilon < 1$ by:

$$\Sigma_\varepsilon(T) := \sigma(T) \bigcup \left\{ \lambda \in \mathbb{C} : \ \|\lambda - T\| \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

with the convention that $\|\lambda - T\| \|(\lambda - T)^{-1}\| = \infty$, if $\lambda - T$ is not invertible.

There are several generalizations of the concept of the spectrum in literature such as Ransford spectrum [11], pseudospectrum [6, 7, 13], condition spectrum [10]. It is natural to ask whether there are any results similar for operator in infinite dimensional Banach spaces for these sets. The concept of condition pseudospectra are interesting objects by themselves since they carry more information than spectra and pseudospectrum, especially, about transient instead of just asymptotic behaviour of dynamical systems. Also, they have better convergence and approximation properties than spectra and pseudospectrum. For a detailed study of the properties of the condition spectrum in finite dimensional space or in Banach algebras, we may refer to S. H. Kulkarni and D. Sukumar in [10], M. Karow in [9] and G. K. Kumar and S. H. Lui in [8].

Our aim in this work is to show some properties of condition pseudospectra of a linear operator T in Banach spaces and reveal the relation between their condition pseudospectrum and the pseudospectrum (resp. numerical range) (Theorem 2.1) (resp. Theorem 3.4). One of the central questions consists in characterizing the condition pseudospectra of all bounded linear operators acting in a Banach space (Theorems 3.2 and 3.3).

The paper is organized as follows. In Section 2, we summarize some properties and useful results of the condition pseudospectrum, in particular, we improve many recent results of [8–10] in infinite dimensional Banach spaces. Then, in Section 3, we will characterize the condition pseudospectrum and we will measure the sensitivity of the set $\sigma(T)$ with taking into consideration to additive perturbations of T by an operator $D \in \mathcal{L}(X)$ of a norm less than $\varepsilon\|\lambda - T\|$.

2. Some properties of $\Sigma_\varepsilon(T)$.

In this section, we define the condition pseudospectrum of linear operators in infinite dimensional Banach spaces and consider some basic properties in order to put this definition in its due place. We begin with the following definition.

Definition 2.1. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. The condition pseudospectrum of T is denoted by $\Sigma_\varepsilon(T)$ and is defined as:*

$$\Sigma_\varepsilon(T) := \sigma(T) \bigcup \left\{ \lambda \in \mathbb{C} : \ \|\lambda - T\| \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

with the convention that $\|\lambda - T\| \|(\lambda - T)^{-1}\| = \infty$, if $\lambda - T$ is not invertible. The condition pseudoresolvent of T is denoted by $\rho_\varepsilon(T)$ and is defined as:

$$\rho_\varepsilon(T) := \rho(T) \bigcap \left\{ \lambda \in \mathbb{C} : \ \|\lambda - T\| \|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon} \right\}.$$

For $0 < \varepsilon < 1$ it can be shown that $\rho_\varepsilon(T)$ is a larger set and is never empty. Here also

$$\rho_\varepsilon(T) \subseteq \rho(T)$$

for $0 < \varepsilon < 1$. Recall that the usual condition pseudospectral radius $r_\varepsilon(T)$ of $T \in \mathcal{L}(X)$ is defined by:

$$r_\varepsilon(T) := \sup \left\{ |\lambda| : \lambda \in \Sigma_\varepsilon(T) \right\},$$

and the spectral radius of $T \in \mathcal{L}(X)$ is defined as:

$$r(T) = \sup \left\{ |\lambda| : \lambda \in \sigma(T) \right\}.$$

In the next Lemma, we give some properties of the condition pseudospectral radius $r_\varepsilon(\cdot)$.

Lemma 2.2. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. Then,*
(*i*) $r(T) \leq r_\varepsilon(T) \leq \Big(\dfrac{1+\varepsilon}{1-\varepsilon}\Big)\|T\|$.
(*ii*) *If* $\|T^k\| \leq \delta < \dfrac{1}{\varepsilon}$ *for all* $k \geq 0$, *then* $r_\varepsilon(T) \leq \dfrac{1+\delta^2\varepsilon}{1-\delta\varepsilon}$.

Proof. (*i*) Since, $\sigma(T) \subseteq \Sigma_\varepsilon(T)$ for $0 < \varepsilon < 1$, we obtain that $r(T) \leq r_\varepsilon(T)$. Now, we prove that

$$r_\varepsilon(T) \leq \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\|T\|.$$

Let $\lambda \in \Sigma_\varepsilon(T)$. We will discuss these two cases:
<u>1^{st}*case*</u> : If $|\lambda| \leq \|T\|$, then it is clear that

$$r_\varepsilon(T) \leq \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\|T\|.$$

<u>2^{nd}*case*</u> : If $|\lambda| > \|T\|$, then $\lambda - T$ is an invertible operator and $\|(\lambda-T)^{-1}\| \leq \dfrac{1}{|\lambda| - \|T\|}$. Since

$$1 < \varepsilon\|(\lambda-T)^{-1}\|\|\lambda - T\| \leq \Big(\frac{|\lambda| + \|T\|}{|\lambda| - \|T\|}\Big)\varepsilon.$$

After a simple computation, we have

$$|\lambda| \leq \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\|T\|.$$

Hence,

$$r_\varepsilon(T) \leq \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\|T\|.$$

(*ii*) Since $\lambda \in \Sigma_\varepsilon(T)$ and $r(T) = \lim\limits_{k\to\infty} \|T^k\|^{\frac{1}{k}} = \lim\limits_{k\to\infty} \delta^{\frac{1}{k}} = 1$, then there are two possible cases:
<u>1^{st}*case*</u> : If $|\lambda| \leq 1$, then it is clear that

$$|\lambda| \leq \frac{1+\delta^2\varepsilon}{1-\delta\varepsilon}.$$

<u>2^{nd}*case*</u> : If $|\lambda| > 1$, then $\lambda \notin \sigma(T)$. We get $\lambda - T$ is an invertible operator. Therefore,

$$\|(\lambda-T)^{-1}\| \leq \frac{1}{|\lambda|}\Big(\sum_{k=0}^{\infty} \frac{\|T^k\|}{|\lambda|^k}\Big) \leq \frac{\delta}{|\lambda|}\Big(\sum_{k=0}^{\infty} \frac{1}{|\lambda|^k}\Big) = \frac{\delta}{|\lambda| - 1}.$$

Hence,

$$1 < \varepsilon\|(\lambda-T)^{-1}\|\|\lambda - T\| \leq \varepsilon(|\lambda| + \delta)\frac{\delta}{|\lambda| - 1},$$

which implies that $|\lambda| \leq \dfrac{1+\delta^2\varepsilon}{1-\delta\varepsilon}$. So, $r_\varepsilon(T) \leq \dfrac{1+\delta^2\varepsilon}{1-\delta\varepsilon}$. □

The following theorem establish the relationship between condition pseudospectrum and pseudospectrum of an bounded linear operator $T \in \mathcal{L}(X)$. We set

$$\delta_T := \inf\Big\{\|\lambda - T\| : \lambda \in \mathbb{C}\Big\}.$$

Theorem 2.3. *Let $T \in \mathcal{L}(X)$ such that $T \neq I$ and $0 < \varepsilon < 1$. Then,*

$$\Sigma_\varepsilon(T) \subseteq \sigma_{\gamma_\varepsilon}(T) \subseteq \Sigma_{\upsilon_\varepsilon}(T)$$

where, $\gamma_\varepsilon = \frac{2\varepsilon\|T\|}{1-\varepsilon}$ *and* $0 < \upsilon_\varepsilon = \frac{2\varepsilon\|T\|}{(1-\varepsilon)\delta_T} < 1$.

Proof. We will prove the first relation of this theorem by the similar ways in [10]. Let $\lambda \in \Sigma_\varepsilon(T)$, then

$$|\lambda| \leq \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)\|T\| \text{ and } \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}.$$

Thus

$$\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon\|\lambda - T\|} > \frac{1}{\varepsilon(|\lambda| + \|T\|)} \geq \frac{1-\varepsilon}{2\varepsilon\|T\|}.$$

Hence $\lambda \in \sigma_{\gamma_\varepsilon}(T)$. For the second inclusion, let $\lambda \in \sigma_{\gamma_\varepsilon}(T)$. Then,

$$\|(\lambda - T)^{-1}\| > \frac{1-\varepsilon}{2\varepsilon\|T\|}.$$

Also, we have $\|\lambda - T\| \geq \inf\Big\{\|\lambda - T\| : \lambda \in \mathbb{C}\Big\} := \delta_T > 0$, hence

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| > \delta_T \frac{1-\varepsilon}{2\varepsilon\|T\|}.$$

Therefore, $\lambda \in \Sigma_{v_\varepsilon}(T)$. □

In the following we gives a precise information about the condition pseudospectrum of bounded linear operator under linear transformation.

Proposition 2.4. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$.*

(i) $\sigma(T) = \bigcap\limits_{0<\varepsilon<1} \Sigma_\varepsilon(T)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2 < 1$, *then* $\sigma(T) \subset \Sigma_{\varepsilon_1}(T) \subset \Sigma_{\varepsilon_2}(T)$.

(iii) If $\alpha \in \mathbb{C}$, *then* $\Sigma_\varepsilon(T + \alpha I) = \alpha + \Sigma_\varepsilon(T)$.

(iv) If $\alpha \in \mathbb{C}\backslash\{0\}$, *then* $\Sigma_\varepsilon(\alpha T) = \alpha\Sigma_\varepsilon(T)$.

Proof. (*i*) It is clear that $\sigma(T) \subset \Sigma_\varepsilon(T)$, then

$$\sigma(T) \subset \bigcap_{0<\varepsilon<1} \Sigma_\varepsilon(T).$$

Conversely, if $\lambda \in \bigcap\limits_{0<\varepsilon<1} \Sigma_\varepsilon(T)$, then for all $0 < \varepsilon < 1$, we have $\lambda \in \Sigma_\varepsilon(T)$. We will discuss these two cases:

<u>1^{st}*case*</u> : If $\lambda \in \sigma(T)$, we get the desired result.

<u>2^{nd}*case*</u> : If $\lambda \in \Big\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \dfrac{1}{\varepsilon}\Big\}$, taking limits as $\varepsilon \to 0^+$, we get

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| = \infty.$$

Thus $\lambda \in \sigma(T)$.

(*ii*) Let $\lambda \in \Sigma_{\varepsilon_1}(T)$, then $\|\lambda - T\|\|(\lambda - T)^{-1}\| > \dfrac{1}{\varepsilon_1} > \dfrac{1}{\varepsilon_2}$. Hence $\lambda \in \Sigma_{\varepsilon_2}(T)$.

(*iii*) Let $\lambda \in \Sigma_\varepsilon(T + \alpha I)$, then $\|(\lambda - \alpha)I - T\|\|(\lambda - \alpha)I - T)^{-1}\| > \dfrac{1}{\varepsilon}$. Hence $\lambda - \alpha \in \Sigma_\varepsilon(T)$. This yields to

$$\lambda \in \alpha + \Sigma_\varepsilon(T).$$

The opposite inclusion follows by symmetry.

(*iv*) Let $\lambda \in \Sigma_\varepsilon(\alpha T)$ such that $\alpha \neq 0$, then we have

$$\begin{aligned}\frac{1}{\varepsilon} < \|\lambda - \alpha T\|\|(\lambda - \alpha T)^{-1}\| &= \|\alpha(\frac{\lambda}{\alpha} - T)\|\|(\alpha^{-1}(\frac{\lambda}{\alpha} - T)^{-1}\|,\\ &= \|(\frac{\lambda}{\alpha} - T)\|\|(\frac{\lambda}{\alpha} - T)^{-1}\|.\end{aligned}$$

Hence $\dfrac{\lambda}{\alpha} \in \Sigma_\varepsilon(T)$, so $\Sigma_\varepsilon(\alpha T) \subseteq \alpha\Sigma_\varepsilon(T)$. However, the opposite inclusion follows by symmetry. □

Theorem 2.5. *Let $T \in \mathcal{L}(X)$, $k = \|T\|\|T^{-1}\|$ and $0 < \varepsilon < 1$. Then,*

(i)$\lambda \in \Sigma_\varepsilon(T)$ if, and only if, $\overline{\lambda} \in \Sigma_\varepsilon(T')$.

(ii) $\lambda \in \Sigma_\varepsilon(T^{-1})\backslash\{0\}$ if, and only if, $\frac{1}{\lambda} \in \Sigma_{\varepsilon k}(T)\backslash\{0\}$.

Proof. (*i*) Using the identity $\|\lambda - T\|\|(\lambda - T)^{-1}\| = \|\overline{\lambda} - T'\|\|(\overline{\lambda} - T')^{-1}\|$, it is easy to see that the condition pseudospectrum of T' is given by the mirror image of $\Sigma_\varepsilon(T)$ with respect to the real axis.

(*ii*) Let $\lambda \in \Sigma_\varepsilon(T^{-1})\backslash\{0\}$, then

$$\begin{aligned}\frac{1}{\varepsilon} < \|\lambda - T^{-1}\|\|(\lambda - T^{-1})^{-1}\| &= \|-\lambda(\frac{1}{\lambda} - T)T^{-1}\|\|-\lambda^{-1}T(\frac{1}{\lambda} - T)^{-1}\|,\\ &\leq \|T\|\|T^{-1}\|\|(\frac{1}{\lambda} - T)\|\|(\frac{1}{\lambda} - T)^{-1}\|.\end{aligned}$$

Hence, $\frac{1}{\lambda} \in \Sigma_{\varepsilon k}(T)\backslash\{0\}$. The converse is similar. □

Theorem 2.6. *Let $T \in \mathcal{L}(X)$ and $V \in \mathcal{L}(X)$ be invertible. Let $B = V^{-1}TV$ Then, for all $0 < \varepsilon < 1$, $k = \|V^{-1}\|\|V\|$ and $0 < k^2\varepsilon < 1$ we have*

$$\Sigma_{\frac{\varepsilon}{k^2}}(T) \subseteq \Sigma_\varepsilon(B) \subseteq \Sigma_{k^2\varepsilon}(T).$$

Proof. We can write

$$\begin{aligned}\|\lambda - B\|\|(\lambda - B)^{-1}\| &= \|V^{-1}(\lambda - T)V\|\|V^{-1}(\lambda - T)^{-1}V\|\\ &\leq \left(\|V\|\|V^{-1}\|\right)^2\|(\lambda - T)\|\|(\lambda - T)^{-1}\|\\ &\leq k^2\|(\lambda - T)\|\|(\lambda - T)^{-1}\|. \qquad (1)\end{aligned}$$

$$\begin{aligned}\|\lambda - T\|\|(\lambda - T)^{-1}\| &= \|V(\lambda - B)V^{-1}\|\|V(\lambda - B)^{-1}V^{-1}\|\\ &\leq \left(\|V\|\|V^{-1}\|\right)^2\|(\lambda - B)\|\|(\lambda - B)^{-1}\|\\ &\leq k^2\|(\lambda - B)\|\|(\lambda - B)^{-1}\|. \qquad (2)\end{aligned}$$

Let $\lambda \in \Sigma_{\frac{\varepsilon}{k^2}}(T)$, then using relation (2), we obtain $\lambda \in \Sigma_\varepsilon(B)$. If $\lambda \in \Sigma_\varepsilon(B)$, then using relation (1), we obtain $\lambda \in \Sigma_{k^2\varepsilon}(B)$. □

3. Characterization of condition pseudospectrum.

In this section, we give a characterization of the condition pseudospectrum of linear operators on a Banach space. Our first result is the following.

Lemma 3.1. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. Then, $\lambda \in \Sigma_\varepsilon(T) \setminus \sigma(T)$ if, and only if, there exists $x \in X$, such that*

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Proof. Let $\lambda \in \Sigma_\varepsilon(T) \setminus \sigma(T)$, then

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}.$$

And thus we have

$$\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Moreover,

$$\sup_{y\in X\setminus\{0\}} \frac{\|(\lambda - T)^{-1}y\|}{\|y\|} > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Then, there exists a nonzero $y \in X$, such that

$$\|(\lambda - T)^{-1}y\| > \frac{\|y\|}{\varepsilon\|\lambda - T\|}.$$

Putting $x = (\lambda - T)^{-1}y$. We have the results. Conversely, we assume there exists $x \in X$ such that

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Let $\lambda \notin \sigma(T)$ and $x = (\lambda - T)^{-1}y$, then $\|x\| \leq \|(\lambda - T)^{-1}\|\|y\|$. Moreover,

$$1 < \varepsilon\|\lambda - T\|\|(\lambda - T)^{-1}\|.$$

So, $\lambda \in \Sigma_\varepsilon(T) \setminus \sigma(T)$. □

In the following theorem, we investigate the relation between the condition pseudospectrum and the usual spectrum in a complex Banach space.

Theorem 3.2. *Let $T \in \mathcal{L}(X)$, $\lambda \in \mathbb{C}$, and $0 < \varepsilon < 1$. If there is $D \in \mathcal{L}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma(T + D)$. Then, $\lambda \in \Sigma_\varepsilon(T)$.*

Proof. We assume that there exists D such that $\|D\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma(T + D)$. Let $\lambda \notin \Sigma_\varepsilon(T)$, then

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}.$$

Now, we define the operator $S : X \longrightarrow X$ by

$$S := \sum_{n=0}^{\infty} (\lambda - T)^{-1}\Big(D(\lambda - T)^{-1}\Big)^n.$$

By the fact that

$$\|D(\lambda - T)^{-1}\| < 1,$$

we can write

$$S = (\lambda - T)^{-1}\Big(I - D(\lambda - T)^{-1}\Big)^{-1}.$$

Then, there exists $y \in X$ such that

$$S\Big(I - D(\lambda - T)^{-1}\Big)y = (\lambda - T)^{-1}y.$$

Let $x = (\lambda - T)^{-1}y$. Then,

$$S(\lambda - T - D)x = x$$

for every $x \in X$. Similarly, we can prove that

$$(\lambda - T - D)Sy = y$$

for all $y \in X$. Hence, $\lambda - T - D$ is invertible, so $\lambda \in \Sigma_\varepsilon(T)$. □

Theorem 3.3. *Suppose X is a complex Banach space with the following property: For all invertible operator $T \in \mathcal{L}(X)$ there exist $D \in \mathcal{L}(X)$ such that D is not invertible and $\|T - D\| = \dfrac{1}{\|T^{-1}\|}$. Then, if $\lambda \in \Sigma_\varepsilon(T)$ there exists $D \in \mathcal{L}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma(T + D)$.*

Proof. Suppose $\lambda \in \Sigma_\varepsilon(T)$. We will discuss these two cases:
1^{st} case : If $\lambda \in \sigma(T)$, then it is sufficient to take $D = 0$.
2^{nd} case : If $\lambda \in \Sigma_\varepsilon(T)\backslash\sigma(T)$. Hence, by assumption, there exists an element $B \in \mathcal{L}(X)$ such that

$$\|\lambda - T - B\| = \frac{1}{\|(\lambda - T)^{-1}\|}.$$

Let $D = \lambda - T - B$. Then

$$\|D\| = \frac{1}{\|(\lambda - T)^{-1}\|} \leq \varepsilon\|\lambda - T\|.$$

Also $B = \lambda - (T + D)$, is not invertible. So, $\lambda \in \sigma(T + D)$. □

Corollary 3.4. *Let X be a complex Banach space satisfying the hypothesis of Theorem 3.3. Then, $\lambda \in \Sigma_\varepsilon(A)$ if, and only if, there exists $D \in \mathcal{L}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma(T + D)$.*

Remark 3.5. *The following Theorem 3.6, refines the above Theorem 3.2 and 3.3.*

Theorem 3.6. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. Then, $\lambda \in \Sigma_\varepsilon(T)$ if, and only if, there exists a bounded operator D such that $\|D\| < \varepsilon\|\lambda - T\|$, $\dim \mathcal{R}(D) < 1$ and $\lambda \in \sigma(T + D)$.*

Proof. Let $\lambda \in \Sigma_\varepsilon(T) := \sigma(T) \bigcup \left\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \dfrac{1}{\varepsilon}\right\}$. We will discuss the two following cases:

1^{st}case : If $\lambda \in \sigma(T)$, we can take $D = 0$.

2^{nd}case : If $\lambda \notin \sigma(T)$, then there exists $x_0 \in X$ such that $\|x_0\| = 1$ and

$$\|(\lambda - T)x_0\| < \varepsilon\|\lambda - T\|.$$

Putting $\|x_0\| = \|(\lambda - T)^{-1}(\lambda - T)x_0\|$ implies that $\|(\lambda - T)^{-1}\| > \dfrac{1}{\varepsilon\|\lambda - T\|}$. Then, we can find $y_0 \in X$ such that $\|y_0\| = 1$ and $\|(\lambda - T)^{-1}y_0\| > \dfrac{1}{\varepsilon\|\lambda - T\|}$. Hence

$$\|(\lambda - T)^{-1}y_0\| = \frac{1}{\delta},$$

where, $0 < \delta < \varepsilon\|\lambda - T\|$. By using the Hahn Banach Theorem, there exists $x' \in X'$ such that $\|x'\| = 1$ and

$$x'\left((\lambda - T)^{-1}y_0\right) = \|(\lambda - T)^{-1}y_0\| = \frac{1}{\delta}.$$

Now, we can define the rank-one operator by,

$$\begin{cases} D : X \longrightarrow X, \\ \qquad x \longrightarrow Dx := \delta x'(x)y_0. \end{cases}$$

Clearly, D is a linear operator everywhere defined on X and bounded, since

$$\|Dx\| = \|\delta x'(x)y_0\| \leq \delta\|x'\|\|y_0\|\|x\|.$$

Then, $\|D\| \leq \delta < \varepsilon\|\lambda - T\|$. Furthermore,

$$D\Big((\lambda - T)^{-1}y_0\Big) = \delta x'\Big((\lambda - T)^{-1}y_0\Big)y_0 = \delta\frac{1}{\delta}y_0 = y_0.$$

Putting $x = (\lambda - T)^{-1}y_0$. If $x = 0$, then $(\lambda - T)^{-1}y_0 = 0$, which is a contradiction with

$$\|(\lambda - T)^{-1}y_0\| = \frac{1}{\delta} > 0.$$

Then, $x \neq 0$, therefore $Dx = y_0 = (\lambda - T)x$. Hence, $\lambda - T - D$ is not injective, so $\lambda \in \sigma(T + D)$. $\square$

Theorem 3.7. *Let $T, E \in \mathcal{L}(X)$ such that $\|E\| < \frac{\varepsilon}{2}\|\lambda - T\|$ and $0 < \varepsilon < 1$. Then,*

$$\Sigma_{\frac{\varepsilon}{2}-\|E\|}(T) \subseteq \Sigma_\varepsilon(T + E) \subseteq \Sigma_{\tau_\varepsilon}(T)$$

where, $0 < \tau_\varepsilon = \dfrac{\varepsilon^2}{2} + \varepsilon < 1$ *and* $0 < \frac{\varepsilon}{2} - \|E\| < 1$.

Proof. Let $\lambda \in \Sigma_{\frac{\varepsilon}{2}-\|E\|}(T)$. Then, by Theorem 3.2, there exists a bounded operator $D \in \mathcal{L}(X)$ with

$$\|D\| < \Big(\frac{\varepsilon}{2} - \|E\|\Big)\|\lambda - T\|$$

such that

$$\lambda \in \sigma(T + D) = \sigma\Big((T + E) + (D - E)\Big).$$

The fact that

$$\begin{array}{rcl} \|D - E\| & \leq & \|D\| + \|E\| \\ & < & \Big(\frac{\varepsilon}{2} - \|E\|\Big)\|\lambda - T\| + \|E\| \\ & < & \varepsilon\|\lambda - T\|, \end{array}$$

allows us to deduce that $\lambda \in \Sigma_\varepsilon(T + E)$. Now, let us prove the second inclusion. Suppose $\lambda \in \Sigma_\varepsilon(T + E)$, then there exists $D \in \mathcal{L}(X)$ verifying

$$\|D\| < \varepsilon\|\lambda - T - E\| \leq \varepsilon\|\lambda - T\| + \varepsilon\|E\|$$

and $\lambda \in \sigma(T + E + D)$. The fact that $\|D + E\| \leq \tau_\varepsilon\|\lambda - T\|$ allows us to deduce that $\lambda \in \Sigma_{\tau_\varepsilon}(T)$. $\square$

In the sequel of this section, we use the notation $conv(\Sigma_\varepsilon(T))$ the convex hull in $\mathbb{C}$ of a set $\Sigma_\varepsilon(T)$, $B(a, r)$ the ball with center at a and radius r and we define the distance between two nonempty subsets U and V by the formula

$$\inf\Big\{\|u - v\| : u \in U,\ v \in V\Big\}.$$

The next theorem gives a relation between the condition pseudospectrum and numerical range on Hilbert space X.

$$\mathcal{W}(T) := \Big\{ < Tx, x >: \|x\| = 1\Big\}.$$

It is well known that $\mathcal{W}(T)$ is a convex set whose closure contains the spectrum $\sigma(T)$ of T.

Theorem 3.8. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. Then,*

$$conv(\Sigma_\varepsilon(T)) \subseteq \mathcal{W}(T) + B(0, \tfrac{2\varepsilon}{1-\varepsilon}\|T\|).$$

Proof. Let $\lambda \in \mathbb{C}$, such that $d(\lambda, \mathcal{W}(T)) > 0$, then $\lambda - T$ is invertible and we have

$$\|(\lambda - T)^{-1}\| \leq \frac{1}{d(\lambda, \mathcal{W}(T))}.$$

Assume that $\lambda \in \Sigma_\varepsilon(T)$. There are two possible cases:

$\underline{1^{st} case}$: If $\lambda \in \overline{\mathcal{W}(T)}$, is trivial.

$\underline{2^{nd} case}$: If $\lambda \in \Sigma_\varepsilon(T)\backslash\overline{\mathcal{W}(T)}$. Then

$$\frac{1}{\varepsilon} < \|\lambda - T\|\|(\lambda - T)^{-1}\| \leq \frac{\|\lambda - T\|}{d(\lambda, \mathcal{W}(T))}.$$

It follows that

$$d(\lambda, \mathcal{W}(T)) < \varepsilon\|\lambda - T\| \leq \varepsilon\,(|\lambda| + \|T\|).$$

Using Lemma 2.2, we obtain that

$$d(\lambda, \mathcal{W}(T)) \leq \varepsilon\left(\frac{1+\varepsilon}{1-\varepsilon}\|T\| + \|T\|\right) = \left(\frac{2\varepsilon}{1-\varepsilon}\right)\|T\|.$$

Hence

$$\Sigma_\varepsilon(T) \subseteq \mathcal{W}(T) + B\Big(0, \frac{2\varepsilon}{1-\varepsilon}\|T\|\Big).$$

By the fact that $\mathcal{W}(T)$ is convex set, we have the results. $\square$

Corollary 3.9. *Let* $T \in \mathcal{L}(X)$ *and* $0 < \varepsilon < 1$. *Then,*

$$\Big\{\lambda \in \mathbb{C} : d(\lambda, \mathcal{W}(T)) < \varepsilon\|\lambda - T\|\Big\} \subseteq \Sigma_\varepsilon(T).$$

Proof. Let $\lambda \notin \Sigma_\varepsilon(T)$, from Theorem 3.8, we have

$$\frac{1}{d(\lambda, \mathcal{W}(T))} \leq \|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon\|\lambda - T\|}.$$

Then, $d(\lambda, \mathcal{W}(T)) \geq \varepsilon\|\lambda - T\|$. $\square$

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