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Weak convergence theorems for two nearly asymptotically nonexpansive non-self mappings

G. S. Saluja[a]

[a] *Department of Mathematics, Govt. N. P. G. College of Science, Raipur - 492010 (C.G.), India.*

Abstract. In this paper, we establish some weak convergence results of modified S-iteration process to converge to common fixed points involving two nearly asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach space under the following conditions (i) the Banach space E satisfying Opial condition and (ii) the dual E^* of E has the Kadec-Klee property. Our results extend and improve many known results from the existing literature.

1. Introduction

Let C be a nonempty subset of a Banach space and $T\colon C \to C$ a nonlinear mapping. We denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$ and $\mathbb{N}$ denotes the set of all positive integers.

The mapping T is said to be Lipschitzian [1, 13] if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\|$$

for all $x, y \in C$.

A Lipschitzian mapping T is said to be uniformly k-Lipschitzian if $k_n = k$ for all $n \in \mathbb{N}$ and asymptotically nonexpansive [7] if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$.

Remark 1.1. *It is easy to observe that every nonexpansive mapping T (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$) is asymptotically nonexpansive with constant sequence $\{1\}$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n\in\mathbb{N}}\{k_n\}$.*

In 2005, Sahu [13] introduced the class of nearly Lipschitzian mappings as an important generalization of the class of Lipschitzian mappings.

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Email address: saluja1963@gmail.com (G. S. Saluja)

Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} a_n = 0$. A mapping T is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exist constants $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n(\|x-y\| + a_n) \tag{1}$$

for all $x,\ y \in C$.

The infimum of constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be

(i) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$ and

(ii) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$.

Example 1.2. *(See [13]) Let $E = \mathbb{R}$, $C = [0,1]$ and $T\colon C \to C$ be a mapping defined by*

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly T is discontinuous and a non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\} = \{\frac{1}{2^n}, 1\}$. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$ as $n \to \infty$, we have

$$\|Tx - Ty\| \leq \|x-y\| + a_1 \ \text{ for all } x, y \in C$$

and

$$\|T^n x - T^n y\| \leq \|x-y\| + a_n \ \text{ for all } x, y \in C \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \ \text{ for all } x \in [0,1] \text{ and } n \geq 2.$$

A subset C of a Banach space E is called a retract of E if there exists a continuous mapping $P\colon E \to C$ (called a retraction) such that $P(x) = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P\colon E \to C$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction then $Py = y$ for all y in the range of P.

In 2003, Chidume et al. [3] defined non-self asymptotically nonexpansive mappings as follows.

Let C be a nonempty subset of a real Banach space E and let $P\colon E \to C$ be a nonexpansive retraction of E onto C. A non-self mapping $T\colon C \to E$ is called asymptotically nonexpansive if there exists a positive sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \quad \leq \quad k_n\|x-y\| \tag{2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Also T is called uniformly k-Lipschitzian if for some $k > 0$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \quad \leq \quad k\|x-y\| \tag{3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

In 2007, Zhou et al. [21] introduced the following definition.

(1) A non-self mapping $T\colon C \to E$ is called asymptotically nonexpansive with respect to P if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|(PT)^n(x) - (PT)^n(y)\| \leq k_n\|x-y\| \tag{4}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(2) T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L > 0$ such that

$$\|(PT)^n(x) - (PT)^n(y)\| \leq L\|x-y\| \tag{5}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Remark 1.3. *([21]) If $T\colon C \to E$ is asymptotically nonexpansive in light of (2) and $P\colon E \to C$ is a nonexpansive retraction, then $PT\colon C \to C$ is asymptotically nonexpansive in light of $\|T^n x - T^n y\| \leq k_n\|x-y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. Indeed by definition (2), we have*

$$\begin{aligned}\|(PT)^n x - (PT)^n y\| &= \|PT(PT)^{n-1}x - PT(PT)^{n-1}y\| \\ &\leq \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \\ &\leq k_n\|x-y\| \end{aligned}\tag{6}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Conversely, it may not be true.

Now, we define the following.

For a sequence $\{a_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} a_n = 0$, then a non-self mapping $T\colon C \to E$ is said to be nearly Lipschitzian with respect to $\{a_n\}$ and P if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|(PT)^n(x) - (PT)^n(y)\| \leq k_n(\|x-y\| + a_n) \tag{7}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. The infimum of constants k_n for which the above inequality holds is, denoted by $\eta((PT)^n)$ and is called nearly Lipschitz constant.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta((PT)^n)\}$ is said to be nearly asymptotically nonexpansive if $\eta((PT)^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta((PT)^n) = 1$.

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1 \end{aligned}\tag{8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions and also they established some weak convergence theorems using suitable conditions in the framework of uniformly convex Banach space.

Chidume et al. [3] studied the following iteration process for non-self asymptotically nonexpansive mappings:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1-\alpha_n)x_n),\ n \geq 1, \end{aligned}\tag{9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Recently, Khan [9] introduced and studied the following iteration scheme for non-self nearly asymptotically nonexpansive mappings defined as:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1-\alpha_n)T(PT)^{n-1}y_n) \\ y_n &= P((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n),\ n \geq 1, \end{aligned} \tag{10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and established some weak convergence theorems under suitable conditions for said mappings in the framework of uniformly convex Banach spaces.

In the light of above we proposed and study the following iteration scheme for two nearly asymptotically nonexpansive non-self mappings $S, T\colon C \to E$ defined as:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P((1-\alpha_n)(PT)^n x_n + \alpha_n (PS)^n y_n) \\ y_n &= P((1-\beta_n)x_n + \beta_n (PT)^n x_n),\ n \geq 1, \end{aligned} \tag{11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

The asymptotic fixed point theory has a fundamental role in nonlinear functional analysis (see [2]). A branch of this theory related to asymptotically nonexpansive self and non-self mappings have been developed by many authors (see, e.g., [2]-[3],[5],[7],[8],[11]-[12],[14]-[16],[20],[21]) in Banach spaces with suitable geometric structure.

The purpose of this paper is to prove some weak convergence theorems of iteration scheme (11) for two nearly asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach spaces.

2. Preliminaries

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

We recall the following.

Let $\mathcal{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E.

The space E has Opial condition [10] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

Definition 2.1. *A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.*

Definition 2.2. *A Banach space E has the Kadec-Klee property [17] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.*

Next we state the following useful lemmas to prove our main results.

Lemma 2.3. *([18]) Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then
(i) $\lim_{n\to\infty} \alpha_n$ exists;
(ii) In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4. *([15]) Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5. *([17]) Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and p, $q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.*

Lemma 2.6. *([17]) Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi\colon [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each Lipschitzian mapping $T\colon C \to C$ with the Lipschitz constant L,*

$$\|tTx + (1-t)Ty - T(tx + (1-t)y)\| \leq L\phi^{-1}\Big(\|x-y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all x, $y \in K$ and all $t \in [0,1]$.

3. Main Results

In this section, we prove some weak convergence theorems for two nearly asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 3.1. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $P\colon E \to C$ is a nonexpansive retraction of E onto C and $S, T\colon C \to E$ be two nearly asymptotically nonexpansive non-self mappings with sequences $\{a'_n, \eta((PS)^n)\}$ and $\{a''_n, \eta((PT)^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \Big[\eta((PS)^n)\eta((PT)^n) - 1\Big] < \infty$. Let $\{x_n\}$ be the iteration scheme defined by (11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$. If $F = F(S) \cap F(T) \neq \emptyset$ and $q \in F$, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in F$. For the sake of simplicity, set

$$D_n(x) = P((1-\beta_n)x + \beta_n(PT)^n x)$$

and

$$R_n(x) = P((1-\alpha_n)(PT)^n x + \alpha_n(PS)^n D_n x).$$

Then $y_n = D_n x_n$ and $x_{n+1} = R_n x_n$. Moreover, it is clear that q is a fixed point of R_n for all n. Let $\eta = \sup_{n\in\mathbb{N}} \eta((PS)^n) \vee \sup_{n\in\mathbb{N}} \eta((PT)^n)$, $a_n = \max\{a'_n, a''_n\}$ for all n, $\lambda_1 = \eta((PS)^n)$ and $\lambda_2 = \eta((PT)^n)$.

Consider

$$\begin{aligned}
\|D_n x - D_n y\| &= \|P((1-\beta_n)x + \beta_n (PT)^n x) \\
&\quad - P((1-\beta_n)y + \beta_n (PT)^n y)\| \\
&\leq \|(1-\beta_n)x + \beta_n (PT)^n x \\
&\quad -(1-\beta_n)y + \beta_n (PT)^n y\| \\
&= \|(1-\beta_n)(x-y) + \beta_n[(PT)^n x - (PT)^n y]\| \\
&\leq (1-\beta_n)\|x-y\| + \beta_n \|(PT)^n x - (PT)^n y\| \\
&\leq (1-\beta_n)\|x-y\| + \beta_n[\eta((PT)^n)(\|x-y\| + a''_n)] \\
&\leq (1-\beta_n)\|x-y\| + \beta_n[\eta((PT)^n)(\|x-y\| + a_n)] \\
&= (1-\beta_n)\|x-y\| + \beta_n \eta((PT)^n)\|x-y\| \\
&\quad +\beta_n \eta((PT)^n) a_n \\
&\leq (1-\beta_n)\eta((PT)^n)\|x-y\| + \beta_n \eta((PT)^n)\|x-y\| \\
&\quad +\beta_n \eta((PT)^n) a_n \\
&\leq \eta((PT)^n)\|x-y\| + \eta((PT)^n) a_n \\
&= \lambda_2 \|x-y\| + \lambda_2 a_n. \qquad (12)
\end{aligned}$$

Choosing $x = x_n$ and $y = q$, we get

$$\|y_n - q\| \leq \lambda_2 \|x-y\| + \lambda_2 a_n. \qquad (13)$$

Now, consider

$$\begin{aligned}
\|R_n x - R_n y\| &= \|P((1-\alpha_n)(PT)^n x + \alpha_n (PS)^n D_n x) \\
&\quad -P((1-\alpha_n)(PT)^n y + \alpha_n (PS)^n D_n y)\| \\
&\leq \|(1-\alpha_n)(PT)^n x + \alpha_n (PS)^n D_n x \\
&\quad -(1-\alpha_n)(PT)^n y + \alpha_n (PS)^n D_n y\| \\
&= \|(1-\alpha_n)[(PT)^n x - (PT)^n y] \\
&\quad +\alpha_n[(PS)^n D_n x - (PS)^n D_n y]\| \\
&\leq (1-\alpha_n)\|(PT)^n x - (PT)^n y\| \\
&\quad +\alpha_n \|(PS)^n D_n x - (PS)^n D_n y\| \\
&\leq (1-\alpha_n)[\eta((PT)^n)(\|x-y\| + a''_n)] \\
&\quad +\alpha_n[\eta((PS)^n)(\|D_n x - D_n y\| + a'_n)] \\
&\leq (1-\alpha_n)[\eta((PT)^n)(\|x-y\| + a_n)] \\
&\quad +\alpha_n[\eta((PS)^n)(\|D_n x - D_n y\| + a_n)] \\
&= (1-\alpha_n)\lambda_2 \|x-y\| + \alpha_n \lambda_1 \|D_n x - D_n y\| \\
&\quad +(1-\alpha_n) a_n \lambda_2 + \alpha_n a_n \lambda_1. \qquad (14)
\end{aligned}$$

Now, using (13) in (14), we get

$$\begin{aligned} \|R_n x - R_n y\| &\leq (1-\alpha_n)\lambda_2\|x-y\| \\ &\quad +\alpha_n\lambda_1[\lambda_2\|x-y\| + \lambda_2 a_n] \\ &\quad +(1-\alpha_n)a_n\lambda_2 + \alpha_n a_n \lambda_1 \\ &\leq [(1-\alpha_n)+\alpha_n]\lambda_1\lambda_2\|x-y\| \\ &\quad +[(1-\alpha_n)+\alpha_n]\lambda_1\lambda_2 a_n \\ &= \lambda_1\lambda_2\|x-y\| + \lambda_1\lambda_2 a_n \\ &= [1+(\lambda_1\lambda_2-1)]\|x-y\| + \lambda_1\lambda_2 a_n \\ &\leq [1+(\lambda_1\lambda_2-1)]\|x-y\| + a_n\eta^2 \\ &= [1+t_n]\|x-y\| + m_n \end{aligned} \tag{15}$$

where $t_n = (\lambda_1\lambda_2 - 1) = \Big(\eta((PS)^n)\eta((PT)^n)\Big) - 1$ and $m_n = a_n\eta^2$. Since by hypothesis $\sum_{n=1}^{\infty}\Big[\eta((PS)^n)\eta((PT)^n) - 1\Big] < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} m_n < \infty$.

Choosing $x = x_n$ and $y = q$ in (15), we get

$$\begin{aligned} \|x_{n+1} - q\| &= \|R_n x_n - q\| \\ &\leq [1+t_n]\|x_n - q\| + m_n. \end{aligned} \tag{16}$$

Applying Lemma 2.3 in (16), we have $\lim_{n\to\infty}\|x_n - q\|$ exists. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $P\colon E \to C$ is a nonexpansive retraction of E onto C and $S, T\colon C \to E$ be two nearly asymptotically nonexpansive non-self mappings with sequences $\{a'_n, \eta((PS)^n)\}$ and $\{a''_n, \eta((PT)^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty}\Big[\eta((PS)^n)\eta((PT)^n) - 1\Big] < \infty$ and $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$. Then $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$ and therefore $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq K = \overline{B_r(0)} \cap C$, so that K is a closed convex subset of C. Let $\lim_{n\to\infty}\|x_n - q\| = c$. Then $c > 0$ otherwise there is nothing to prove.

Now (13) implies that

$$\limsup_{n\to\infty}\|y_n - q\| \leq c. \tag{17}$$

Also

$$\begin{aligned} \|(PT)^n x_n - q\| &\leq \eta((PT)^n)[\|x_n - q\| + a''_n] \\ &\leq \eta((PT)^n)[\|x_n - q\| + a_n], \end{aligned}$$

for all $n = 1, 2, \ldots$ and

$$\begin{aligned} \|(PS)^n x_n - q\| &\leq \eta((PS)^n)[\|x_n - q\| + a'_n] \\ &\leq \eta((PS)^n)[\|x_n - q\| + a_n], \end{aligned}$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty}\|(PT)^n x_n - q\| \leq c. \tag{18}$$

$$\limsup_{n\to\infty}\|(PS)^n x_n - q\| \leq c. \tag{19}$$

Next

$$\begin{aligned}\|(PS)^n y_n - q\| &\leq \eta((PS)^n)[\|y_n - q\| + a'_n] \\ &\leq \eta((PS)^n)[\|y_n - q\| + a_n]\end{aligned}$$

gives by virtue of (17) that

$$\limsup_{n\to\infty} \|(PS)^n y_n - q\| \leq c. \tag{20}$$

Since

$$\begin{aligned}c &= \lim_{n\to\infty} \|x_{n+1} - q\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(PT)^n x_n + \alpha_n (PS)^n y_n - q\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)[(PT)^n x_n - q] + \alpha_n[(PS)^n y_n - q]\|\end{aligned}$$

and Lemma 2.4 that

$$\lim_{n\to\infty} \|(PT)^n x_n - (PS)^n y_n\| = 0. \tag{21}$$

From (11) and (21), we have

$$\begin{aligned}\|x_{n+1} - (PT)^n x_n\| &= \alpha_n \|(PT)^n x_n - (PS)^n y_n\| \\ &\leq (1-\delta)\|(PT)^n x_n - (PS)^n y_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{22}$$

Hence

$$\begin{aligned}\|x_{n+1} - (PS)^n y_n\| &\leq \|x_{n+1} - (PT)^n x_n\| \\ &\quad + \|(PT)^n x_n - (PS)^n y_n\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{23}$$

Next

$$\begin{aligned}\|x_{n+1} - q\| &\leq \|x_{n+1} - (PS)^n y_n\| + \|(PS)^n y_n - q\| \\ &\leq \|x_{n+1} - (PS)^n y_n\| + \eta((PS)^n)[\|y_n - q\| + a'_n] \\ &\leq \|x_{n+1} - (PS)^n y_n\| + \eta((PS)^n)[\|y_n - q\| + a_n]\end{aligned} \tag{24}$$

which gives from (23) that

$$c \leq \liminf_{n\to\infty} \|y_n - q\|. \tag{25}$$

From (17) and (25), we obtain

$$\begin{aligned}c &= \lim_{n\to\infty} \|y_n - q\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n (PT)^n x_n - q\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)[x_n - q] + \beta_n[(PT)^n x_n - q]\|\end{aligned}$$

and it follows from Lemma 2.4 that

$$\lim_{n\to\infty} \|x_n - (PT)^n x_n\| = 0. \tag{26}$$

Again note that

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)x_n + \beta_n (PT)^n x_n - x_n\| \\ &= \beta_n \|x_n - (PT)^n x_n\| \\ &\leq (1-\delta)\|x_n - (PT)^n x_n\| \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned} \tag{27}$$

Also, from (21) and (26), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|P((1-\alpha_n)(PT)^n x_n + \alpha_n (PS)^n y_n) - P(x_n)\| \\ &\leq \|(1-\alpha_n)(PT)^n x_n + \alpha_n (PS)^n y_n - x_n\| \\ &= \|x_n - (PT)^n x_n\| + \alpha_n \|(PT)^n x_n - (PS)^n y_n\| \\ &\leq \|x_n - (PT)^n x_n\| + (1-\delta)\|(PT)^n x_n - (PS)^n y_n\| \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned} \tag{28}$$

Thus, from (27) and (28), we obtain

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - y_n\| \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned} \tag{29}$$

Furthermore, we have

$$\begin{aligned} \|x_{n+1} - (PT)^n y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - (PT)^n x_n\| \\ &\quad + \|(PT)^n x_n - (PT)^n y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_n - (PT)^n x_n\| \\ &\quad + \eta((PT)^n)[\|x_n - y_n\| + a''_n] \\ &\leq \|x_{n+1} - x_n\| + \|x_n - (PT)^n x_n\| \\ &\quad + \eta((PT)^n)[\|x_n - y_n\| + a_n]. \end{aligned} \tag{30}$$

Using (26), (27), (28) and $a_n \to 0$ as $n \to \infty$ in (30), we get

$$\lim_{n\to\infty} \|x_{n+1} - (PT)^n y_n\| = 0. \tag{31}$$

Finally, we make use of the fact that every nearly asymptotically nonexpansive mapping is nearly k-Lipschitzian, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - (PT)^n x_n\| + \|(PT)^n x_n - (PT)^n y_{n-1}\| \\ &\quad + \|(PT)^n y_{n-1} - Tx_n\| \\ &= \|x_n - (PT)^n x_n\| + \|(PT)^n x_n - (PT)^n y_{n-1}\| \\ &\quad + \|(PT)(PT)^{n-1} y_{n-1} - (PT)x_n\| \\ &\leq \|x_n - (PT)^n x_n\| + \eta((PT)^n)[\|x_n - y_{n-1}\| + a''_n] \\ &\quad + k\|(PT)^{n-1} y_{n-1} - x_n\| \\ &\leq \|x_n - (PT)^n x_n\| + \eta((PT)^n)[\|x_n - y_{n-1}\| + a_n] \\ &\quad + k\|(PT)^{n-1} y_{n-1} - x_n\|. \end{aligned} \tag{32}$$

Using (26), (29), (30) and $a_n \to 0$ as $n \to \infty$ in (32), we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{33}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{34}$$

This completes the proof. □

Theorem 3.3. *Let E be a uniformly convex Banach space satisfying Opial's condition and C be a nonempty closed convex subset of E. Let $P: E \to C$ is a nonexpansive retraction of E onto C and $S, T: C \to E$ be two nearly asymptotically nonexpansive non-self mappings with sequences $\{a'_n, \eta((PS)^n)\}$ and $\{a''_n, \eta((PT)^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} \left[\eta((PS)^n)\eta((PT)^n) - 1\right] < \infty$ and $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$. If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $q \in F$, from Lemma 3.1 the sequence $\{\|x_n - q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q^* \in C$. From Lemma 3.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0 \text{ and } \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

Since the mappings $I-S$ and $I-T$ are demiclosed at zero, therefore $Sq^* = q^*$ and $Tq^* = q^*$ which means $q^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to $q^* \in F$. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p^* \in C$ and $q^* \neq p^*$. Then by the same method as given above, we can also prove that $p^* \in F$. From Lemma 3.1 the limits $\lim_{n\to\infty} \|x_n - q^*\|$ and $\lim_{n\to\infty} \|x_n - p^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q^*\| &= \lim_{n_k\to\infty} \|x_{n_k} - q^*\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - p^*\| \\
&= \lim_{n\to\infty} \|x_n - p^*\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - p^*\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - q^*\| \\
&= \lim_{n\to\infty} \|x_n - q^*\|
\end{aligned}$$

which is a contradiction, so $q^* = p^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of S and T. This completes the proof. □

It is well known that there exist classes of uniformly convex Banach spaces without the Opial condition (e.g., L_p spaces $p \neq 2$). Therefore, Theorem 3.3 is not true for such Banach spaces. We now show that Theorem 3.3 is valid if the assumption that E satisfies the Opial condition is replaced by the dual E^* of E has the Kadec-Klee property (KK-property).

Example 3.4. *(Example 3.1, [6]) Let us take $X_1 = \mathbb{R}^2$ with the norm denoted by $\|x\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$ and $X_2 = L_p[0,1]$ with $1 < p < \infty$ and $p \neq 2$. The Cartesian product of X_1 and X_2 furnished with the ℓ^2-norm is uniformly convex, it does not satisfy the Opial condition but its dual does have the Kadec-Klee property (KK-property).*

Lemma 3.5. *Under the assumptions of Lemma 3.2, for all p, $q \in F$, the limit*

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$, where $\{x_n\}$ is the sequence defined by (11).

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F$ and therefore $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 3.5 for $t \in (0,1)$. For all $x \in C$, we define the mapping $R_n: C \to C$ by:

$$D_n(x) = P((1-\beta_n)x + \beta_n(PT)^n x)$$

and

$$R_n(x) = P((1-\alpha_n)(PT)^n x + \alpha_n (PS)^n D_n(x)).$$

Then it follows that $x_{n+1} = R_n x_n$, $R_n p = p$ for all $p \in F$ and we have shown earlier in Lemma 3.1 that

$$\begin{aligned} \|R_n(x) - R_n(y)\| &\leq (1+t_n)\|x-y\| + m_n \\ &= \mu_n \|x-y\| + m_n \end{aligned} \tag{35}$$

for all $x, y \in C$, where $t_n = (\lambda_1\lambda_2 - 1) = \big(\eta((PS)^n)\eta((PT)^n)\big) - 1$ and $m_n = a_n\eta^2$ with $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} m_n < \infty$, $\mu_n = 1 + t_n$ and $\mu_n \to 1$ as $n \to \infty$. Setting

$$U_{n,m} = R_{n+m-1}R_{n+m-2}\dots R_n, \ m \geq 1 \tag{36}$$

and

$$b_{n,m} = \|U_{n,m}(tx_n + (1-t)p) - (tU_{n,m}x_n + (1-t)U_{n,m}q)\|.$$

From (35) and (36), we have

$$\begin{aligned} \|U_{n,m}(x) - U_{n,m}(y)\| &= \|R_{n+m-1}R_{n+m-2}\dots R_n(x) - R_{n+m-1}R_{n+m-2}\dots R_n(y)\| \\ &\leq \mu_{n+m-1}\|R_{n+m-2}\dots R_n(x) - R_{n+m-2}\dots R_n(y)\| \\ &\quad + m_{n+m-1} \\ &\leq \mu_{n+m-1}\mu_{n+m-2}\|R_{n+m-3}\dots R_n(x) - R_{n+m-3}\dots R_n(y)\| \\ &\quad + m_{n+m-1} + m_{n+m-2} \\ &\ \vdots \\ &\leq \Big(\prod_{j=n}^{n+m-1} \mu_j\Big)\|x-y\| + \sum_{j=n}^{n+m-1} m_j \\ &= L_n\|x-y\| + \sum_{j=n}^{n+m-1} m_j \end{aligned} \tag{37}$$

for all $x, y \in C$, where $L_n = \prod_{j=n}^{n+m-1} \mu_j$ and $U_{n,m}x_n = x_{n+m}$ and $U_{n,m}z = z$ for all $z \in F$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,m} + \|U_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m} + L_n a_n(t) + \sum_{j=n}^{n+m-1} m_j. \end{aligned} \tag{38}$$

By using [[6], Theorem 2.3], we have

$$\begin{aligned} b_{n,m} &\leq \varphi^{-1}(\|x_n - u\| - \|U_{n,m}x_n - U_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - U_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|U_{n,m}u - u\|)) \end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, i.e., $b_{n,m} \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} L_n = 1$ and $\sum_{n=1}^{\infty} m_n < \infty$, that is, $\lim_{n\to\infty} m_n = 0$, therefore from (38), we have

$$\limsup_{n\to\infty} a_n(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is, $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. This completes the proof. □

Now, we prove a weak convergence theorem for the spaces whose dual have Kadec-Klee (KK-property).

Theorem 3.6. *Let E be a uniformly convex Banach space such that the dual E^* has the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $P\colon E \to C$ is a nonexpansive retraction of E onto C and $S, T\colon C \to E$ be two nearly asymptotically nonexpansive non-self mappings with sequences $\{a'_n, \eta((PS)^n)\}$ and $\{a''_n, \eta((PT)^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty}\Big[\eta((PS)^n)\eta((PT)^n) - 1\Big] < \infty$ and $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$. If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. By Lemma 3.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in C$. By Lemma 3.2, we have

$$\lim_{j\to\infty} \|x_{n_j} - Sx_{n_j}\| = 0 \text{ and } \lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0$$

Since by hypothesis the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sp = p$ and $Tp = p$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in C$. By the same method as above, we have $q \in F$ and $p, q \in W_w(\{x_n\})$. By Lemma 3.5, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 2.5. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

Remark 3.7. *([4]) It is well known that duals of reflexive Banach spaces with Frěchet differentiable norm have the Kadec-Klee property. However, it is worth mentioning that there exist uniformly convex Banach spaces which have neither a Frěchet differentiable norm nor satisfy Opial's condition but their dual do have the Kadec-Klee property.*

Remark 3.8. *The dual E^* of E has the Kadec-Klee property, Theorem 3.6 generalizes Theorem 2.1 of Schu [16] to the case of two non-self mappings and modified S-iteration scheme in Banach spaces that includes L_p-spaces $(1 < p < \infty)$, with Opial's condition and boundedness of C dispensed with. Since duals of reflexive Banach spaces with Frěchet differentiable norm have the Kadec-Klee property, Theorem 3.6 extends Theorem 3.1 of Tan and Xu [19] to the case of non-self mappings which are nearly asymptotically nonexpansive and modified S-iteration scheme, with boundedness of C dispensed with.*

Example 3.9. *Let E be the real line with the usual norm $|.|$, $C = [0,1]$ and P be the identity mapping. Assume that $S(x) = x$ and $T(x) = \frac{x+1}{2}$ for all $x \in C$. Let $\{a'_n\}_{n\geq 1}$ and $\{a''_n\}_{n\geq 1}$ be two nonnegative real sequences defined by $a'_n = \frac{1}{2^n}$ and $a''_n = \frac{1}{3^n}$ for all $n \geq 1$ with $a'_n \to 0$ and $a''_n \to 0$ as $n \to \infty$. Then S and T are nearly nonexpansive mappings and hence are nearly asymptotically nonexpansive mappings with common fixed point 1, that is, $F = F(S) \cap F(T) = \{1\}$.*

4. Concluding remarks

In this note, we establish some weak convergence theorems for newly defined modified two-step iteration scheme for two nearly asymptotically nonexpansive non-self mappings, this class of mappings is larger than the class of nonexpansive and asymptotically nonexpansive mappings, in the framework of uniformly convex Banach spaces. The results presented in this note extend, generalize and improve the previous works from the existing literature.

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