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Coupled fixed points results for w-compatible mappings in symmetric G-metric spaces

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Abstract. Mustafa et. al [19] generalized the concept of metric space by introducing G-metric space and proved fixed point theorems for mappings satisfying different contractive conditions (see [15–21]). In this paper, we introduce the notion of w-compatible mappings, b-coupled coincidence points and b-common coupled fixed points for non self mappings and obtain fixed point results using these new notions in G-metric spaces. It is worth to mention that our results neither rely on completeness of the space nor the continuity of any mappings involved therein. Also, relevant examples have been cited to illustrate the effectiveness of our results. As an application, we demonstrate the existence of solution of system of non linear integral equations. Our work sets analogues, unifies, generalizes, extends and improves several well known results existing in literature, in particular the recent results of [1, 2, 8–11, 24, 25, 27] etc. in the frame work of G-metric spaces.

1. Introduction

In 2004, Mustafa et al. [19] introduced the concept of G-metric spaces as a generalization of the metric spaces. In this type of space a non-negative real number is assigned to every triplet of elements. In [16], Banach contraction principle was established. After that several fixed point results have been proved in this space. Some of these works may be noted in [1, 9, 15–21, 23, 25]. Coupled fixed point problems belong to a category of problems in fixed point theory in which much interest has been generated recently after the publication of a coupled contraction theorem by Bhaskar and Lakshmikantham [5]. One of the reason for this interest is the application of these results for proving the existence and uniqueness of the solution of differential equations, integral equations, voltra integral and fredoholm integral equations and boundary value problems. For comprehensive description of such work , we refer to [4–7, 13, 23, 24, 28–30]. Later on these results were extended and generalized in [4, 13, 22, 23, 28, 29].

Jungck [10, 11] introduced the concept of commuting and compatible mappings in metric spaces. Since then, these concepts were exploited by many authors [1–4, 8–12, 25–27] to prove a multitude of results of varied kind. In [13], we establish the notion of weakly compatible for coupled mappings and proved

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coupled coincidence and coupled fixed point results and for our work we refer to [6, 7, 13, 28–30]. The intent of this paper is to introduce the concept of w-compatible mappings, b-coupled coincidence points and b-common coupled fixed points for non-self mappings and obtain fixed point results using these new notions in symmetric G-metric spaces. Now we give some preliminaries and basic definitions which are used throughout the paper.

2. Preliminaries

In 2004, Z. Mustafa et al. [21] introduced the concept of G-metric spaces as follows:

Definition 2.1. *Let X be a non-empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following properties:*

(G-1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G-2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

(G-3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ *(symmetry in all three variables),*

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$, *(rectangle inequality).*

The function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X, G) is called a G-metric space. If condition (G-6) is also satisfied then (X, G) is called Symmetric G-metric space.

(G-6) $G(x, x, y) = G(x, y, y)$.

For more details on G-metric spaces we refer to [6, 9, 13, 28].

Definition 2.2. *Let (X, G) be a G-metric space. A point x in X is called a coincidence point of self mappings f and g on X if $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.*

Definition 2.3. *A pair of self mappings (f, g) of a G-metric space (X, G) is said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some u in X, then $fgu = gfu$.*

Definition 2.4. *An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $f : X \times X \to X$ and $g : X \to X$ if $f(x, y) = g(x)$, $f(y, x) = g(y)$.*

Definition 2.5. *An element $(x, y) \in X \times X$ is called*

1. *a common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, y) = g(x)$, $y = f(y, x) = g(y)$.*
2. *a common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, x) = g(x)$.*

Definition 2.6. *The mappings $f : X \times X \to X$ and $g : X \to X$ are said to be weakly compatible if $gf(x, y) = f(gx, gy)$ whenever $f(x, y) = gx$ and $gf(y, x) = f(gy, gx)$ whenever $f(y, x) = gy$.*

Example 2.7. *Let $X = \mathbb{R}$ and $f : X \times X \to X$ and $g : X \to X$ be mappings defined as $f(x, y) = x^2 + y^2$, $g(x) = 2x$, then (1,1) and (0,0) are coupled coincidence point of f and g but the mappings are not weakly compatible as $gf(1, 1) = g(2) = 4 \neq f(g(1), g(1)) = f(2, 2) = 8$.*

Example 2.8. *Let $X = \mathbb{R}$ and $f : X \times X \to X$ and $g : X \to X$ be mappings defined as $f(x, y) = 3x + 2y - 6$, $g(x) = x$, then (1,2) and (2,1) are coupled coincidence point of f and g as $f(1, 2) = 1 = g(1)$ and $f(2, 1) = 2 = g(2)$. Also, $gf(x, y) = g(3x + 2y - 6) = 3x + 2y - 6 = f(g(x), g(y)) = f(x, y)$, which shows that f and g are weakly compatible.*

Definition 2.9. *An element $(x, y) \in X \times X$ is called a b-coupled coincidence point of the mappings $f, g : X \times X \to X$ if $f(x, y) = g(x, y)$, $f(y, x) = g(y, x)$.*

Definition 2.10. *An element $(x, y) \in X \times X$ is called a b-common coupled fixed point of the mappings $f, g : X \times X \to X$ if $x = f(x, y) = g(x, y)$, $y = f(y, x) = g(y, x)$.*

Example 2.11. *Let $X = \mathbb{R}$ and $f, g : X \times X \to X$ be mappings defined as $f(x, y) = 3x - 2y + 1$, $g(x, y) = 2x - 3y + 2$, then $f(1, 0) = 4 = g(1, 0)$ and $f(0, 1) = 1 = g(0, 1)$. Hence (1,0) is a b-coupled coincidence point of f and g and (4,-1) is a b-coupled point of coincidence.*

Example 2.12. *Let $X = \mathbb{R}$ and $f, g : X \times X \to X$ be mappings defined as*

$$f(x, y) = \begin{cases} x + y - 2, & \text{if } x < y; \\ x - y + 1, & \text{if } x \geq y. \end{cases} \quad g(x, y) = \begin{cases} 2x - y + 1, & \text{if } x < y; \\ y - 2x + 5, & \text{if } x \geq y. \end{cases}$$

$f(1, 2) = 1 = g(1, 2)$ and $f(2, 1) = 2 = g(2, 1)$. Hence (1,2) is a b-common coupled fixed point of f and g.

Definition 2.13. *The mappings $f, g : X \times X \to X$ are called weakly compatible if $f(g(x, y), g(y, x)) = g(f(x, y), f(y, x))$ whenever $f(x, y) = g(x, y)$ and $f(y, x) = g(y, x)$.*

Example 2.14. *Let $X = \mathbb{R}$ and $f, g : X \times X \to X$ be mappings defined as $f(x, y) = x + y$, $g(x, y)x - y$, then (x, y) is a b-coupled coincidence point of f and g if and only if $x = y$. Moreover, we have $f(g(x, x), g(x, x)) = g(f(x, x), f(x, x))$ for all $x \in X$ and hence f and g are weakly compatible.*

3. Main results

Theorem 3.1. *Let (X, G) be a symmetric G-metric space and $A, B : X \times X \to X$ be mappings satisfying the following conditions:*

1. *$A(X \times X) \subseteq B(X \times X)$;*
2. *$\{B(x, y), B(y, x)\}$ is a complete subspace of $X \times X$;*
3. $$\begin{aligned} G(A(x, y), A(u, v), A(u, v)) \leq\ & \alpha G(A(x, y), B(u, v), B(u, v)) + \beta G(B(x, y), A(u, v), A(u, v)) \\ & + \gamma G(B(x, y), B(u, v), B(u, v)) + \delta G(A(u, v), B(x, y), B(x, y)) \end{aligned} \tag{1}$$

   $\forall x, y, u, v \in X$, $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 2\beta + \gamma + 2\delta < 1$. Then A and B have a b-coupled coincidence point $(x, y) \in X \times X$, i.e., $A(x, y) = B(x, y)$ and $A(y, x) = B(y, x)$.
   Moreover if the pair (A, B) is weakly compatible, then there exists unique $x \in X$ such that $A(x, x) = B(x, x)$.

Proof. Let x_0, y_0 be two arbitrary points in X. Since $A(X \times X) \subseteq B(X \times X)$, we can choose x_1, y_1 in X such that $A(x_0, y_0) = B(x_1, y_1)$ and $A(y_0, x_0) = B(y_1, x_1)$. Continuing in this way, we can construct two sequences $\{z_n\}$ and $\{t_n\}$ in X such that

$$z_{2n} = A(x_{2n}, y_{2n}) = B(x_{2n+1}, y_{2n+1}), t_{2n} = A(y_{2n}, x_{2n}) = B(y_{2n+1}, x_{2n+1}), \forall n \geq 0.$$

Step 1. We first show that $\{z_n\}$ and $\{t_n\}$ are Cauchy sequences. Using (1)

$$\begin{aligned} G(z_{2n}, z_{2n+1}, z_{2n+1}) &= G(A(x_{2n}, y_{2n}), A(x_{2n+1}, y_{2n+1}), A(x_{2n+1}, y_{2n+1})) \\ &\leq \alpha G(A(x_{2n}, y_{2n}), B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1})) \\ &\quad + \beta G(B(x_{2n}, y_{2n}), A(x_{2n+1}, y_{2n+1}), A(x_{2n+1}, y_{2n+1})) \\ &\quad + \gamma G(B(x_{2n}, y_{2n}), B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1})) \\ &\quad + \delta G(A(x_{2n+1}, y_{2n+1}), B(x_{2n}, y_{2n}), B(x_{2n}, y_{2n}) \\ &= \alpha G(z_{2n}, z_{2n}, z_{2n}) + \beta G(z_{2n-1}, z_{2n+1}, z_{2n+1}) + \gamma G(z_{2n-1}, z_{2n}, z_{2n}) \\ &\quad + \delta G(z_{2n-1}, z_{2n+1}, z_{2n+1}) \\ &\leq 0 + \beta G(z_{2n-1}, z_{2n}, z_{2n}) + \beta G(z_{2n}, z_{2n+1}, z_{2n+1}) \\ &\quad + \gamma G(z_{2n-1}, z_{2n}, z_{2n}) + \delta G(z_{2n-1}, z_{2n}, z_{2n}) + \delta G(z_{2n}, z_{2n+1}, z_{2n+1}), \end{aligned}$$

it implies

$$\begin{aligned}(1-\beta-\delta)G(z_{2n},z_{2n+1},z_{2n+1}) &\leq (\beta+\gamma+\delta)G(z_{2n-1},z_{2n},z_{2n})\\ G(z_{2n},z_{2n+1},z_{2n+1}) &\leq kG(z_{2n-1},z_{2n},z_{2n}),\end{aligned}$$

where $k=\left(\frac{\beta+\gamma+\delta}{1-\beta-\delta}\right)$.
Similarly, we can have $G(z_{2n+1},z_{2n+2},z_{2n+2}) \leq kG(z_{2n},z_{2n+1},z_{2n+1})$. In general,

$$G(z_n,z_{n+1},z_{n+1}) \leq kG(z_{n-1},z_n,z_n) \leq k^2G(z_{n-2},z_{n-1},z_{n-1}) \leq \ldots \leq k^nG(z_0,z_1,z_1).$$

Therefore $\forall n,m \in \mathbb{N}$, $n<m$

$$\begin{aligned}G(z_n,z_m,z_m) &\leq G(z_n,z_{n+1},z_{n+1})+G(z_{n+1},z_{n+2},z_{n+2})+\ldots+G(z_{m-1},z_m,z_m)\\ &\leq (k^n+k^{n+1}+k^{n+2}+\ldots+k^{m-1})G(z_0,z_1,z_1)\end{aligned}$$

Thus, $G(z_n,z_m,z_m)\to 0$ as $n,m\to\infty$. Similarly, we can show that $G(z_m,z_n,z_n)\to 0$ $n,m\to\infty$ and hence $\{z_n\}$ is a Cauchy sequence in X. Similar, we can show that $\{t_n\}$ is a Cauchy sequence. On the other hand, we have $\{z_n,t_n\}=\{B(x_n,y_n),B(y_n,x_n)\}\in\{B(x,y),B(y,x);x,y\in X\}$ is a complete subspace of $X\times X$, so $\exists(z,t)\in\{B(x,y),B(y,x);x,y\in X\}$ such that $G(z_n,t_n)\to G(z,t)$. It implies $\exists(x,y),(y,x)\in X\times X$ such that $z=B(x,y)$ and $t=B(y,x)$ with $z_n\to z=B(x,y)$ and $t_n\to t=B(y,x)$ as $n\to\infty$. From (1), we have

$$\begin{aligned}G(z_n,A(x,y),A(x,y)) &= G(A(x_n,y_n),A(x,y),A(x,y))\\ &\leq \alpha G(A(x_n,y_n),B(x,y),B(x,y))+\beta G(B(x_n,y_n),A(x,y),A(x,y))\\ &\quad +\gamma G(B(x_n,y_n),B(x,y),B(x,y))+\delta G(A(x,y),B(x_n,y_n),B(x_n,y_n)).\end{aligned}$$

Taking limit as $n\to\infty$, we get

$$\begin{aligned}G(z,A(x,y),A(x,y)) &\leq \alpha G(z,B(x,y),B(x,y))+\beta G(z,A(x,y),A(x,y))+\gamma G(z,B(x,y),B(x,y))\\ &\quad +\delta G(A(x,y),z,z),\end{aligned}$$

and so

$$\begin{aligned}G(B(x,y),A(x,y),A(x,y)) &\leq \alpha G(B(x,y),B(x,y),B(x,y))+\beta G(B(x,y),A(x,y),A(x,y))\\ &\quad +\gamma G(B(x,y),B(x,y),B(x,y))+\delta G(A(x,y),B(x,y),B(x,y))\\ (1-\beta)G(B(x,y),A(x,y),A(x,y)) &\leq \delta G(A(x,y),B(x,y),B(x,y))\\ G(B(x,y),A(x,y),A(x,y)) &\leq \frac{\delta}{1-\beta}G(A(x,y),B(x,y),B(x,y)),\end{aligned}$$

or, equivalently

$$\begin{aligned}G(B(x,y),A(x,y),A(x,y)) &\leq qG(A(x,y),B(x,y),B(x,y))\\ &= qG(B(x,y),A(x,y),A(x,y)),\end{aligned}$$

where $q=\frac{\delta}{1-\beta}$ and hence $A(x,y)=B(x,y)$. Similarly, we have $A(y,x)=B(y,x)$ and hence (x,y) is a coincidence point of the mappings A and B.
Step 2. Let the pair (A,B) be weakly compatible and $A(x,y)=B(x,y)$ and $A(y,x)=B(y,x)$, so

$$A(B(x,y),B(y,x))=B(A(x,y),A(y,x))\Rightarrow A(z,t)=B(z,t)$$

and

$$A(B(y,x),B(x,y))=B(A(y,x),A(x,y))\Rightarrow A(t,z)=B(t,z).$$

Using (1), we have

$$\begin{aligned} G(A(B(x_n,y_n),B(y_n,x_n)),A(x_n,y_n),A(x_n,y_n)) &\le \alpha G(A(B(x_n,y_n),B(y_n,x_n)),B(x_n,y_n),B(x_n,y_n)) \\ &\quad +\beta G(B(B(x_n,y_n),B(y_n,x_n)),A(x_n,y_n),A(x_n,y_n)) \\ &\quad +\gamma G(B(B(x_n,y_n),B(y_n,x_n)),B(x_n,y_n),B(x_n,y_n)) \\ &\quad +\delta G(A(x_n,y_n),B(B(x_n,y_n),B(y_n,x_n)),B(B(x_n,y_n),B(y_n,x_n))) \\ G(A(z_n,t_n),z_n,z_n) &\le \alpha G(A(z_n,t_n),z_n,z_n)+\beta G(B(z_n,t_n),z_n,z_n) \\ &\quad +\gamma G(B(z_n,t_n),z_n,z_n)+\delta G(z_n,B(z_n,t_n),B(z_n,t_n)). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$G(A(z,t),z,z) \le \alpha G(A(z,t),z,z)+\beta G(A(z,t),z,z)+\gamma G(A(z,t),z,z)+\gamma G(z,A(z,t),A(z,t)).$$

Which yields, $A(z,t)=z=B(z,t)$. Similarly, we have $A(t,z)=t=B(t,z)$.
Step 3. Now we claim $z=t$. Again from (1), we have

$$\begin{aligned} G(A(x_n,y_n),A(y_n,x_n),A(y_n,x_n)) &\le \alpha G(A(x_n,y_n),B(y_n,x_n),B(y_n,x_n))+\beta G(B(x_n,y_n),A(y_n,x_n),A(y_n,x_n)) \\ &\quad +\gamma G(B(x_n,y_n),B(y_n,x_n),B(y_n,x_n))+\delta G(A(y_n,x_n),B(x_n,y_n),B(x_n,y_n)) \\ G(z_n,t_n,t_n) &\le (\alpha+\beta+\gamma+\delta)G(z_n,t_n,t_n). \end{aligned}$$

Taking $n \to \infty$, we get $z=t$ and hence the result follows. □

Example 3.2. *Let $X=\mathbb{R}$ and G be the G-metric defined on $X\times X\times X \to \mathbb{R}$ defined as $G(x,y,z)=|x-y|+|y-z|+|z-x|$. Then (X,G) is a G-metric space. Define the mappings $A,B : X\times X \to X$ as follows:*

$$A(x,y)=\begin{cases} \frac{x-y}{5}, & x,y\in[0,1]; \\ 1, & otherwise. \end{cases} \quad B(x,y)=\begin{cases} x-y, & x,y\in[0,2]; \\ 1, & otherwise. \end{cases}$$

Then all the conditions of Theorem 3.1 are satisfied. For all $x,y,u,v \in X$, we have

$$G(A(x,y),A(u,v),A(u,v))=\frac{2}{5}(|x-y|-|u-v|)=\frac{2}{5}G(B(x,y),B(u,v),B(u,v)).$$

Thus the contractive condition is satisfied with $\alpha=\beta=\delta=0$, $\gamma=\frac{2}{5}$ and consequently A and B have a b-coupled coincidence point.

In this case, for any $x,y\in[0,1]$, (x,y) is a b-coupled coincidence point if and only if $x=y$. Moreover, we have $A(B(x,y),B(y,x))=A(0,0)=0=B(A(x,y),A(y,x))$.

Thus A and B are weakly compatible mappings, so by Theorem 3.1, we obtain the existence and uniqueness of b-common coupled fixed point of A and B and here (0,0) is the unique b-common coupled fixed point.

Theorem 3.3. *Let (X,G) be a symmetric G-metric space and $A : X\times X \to X$ and $S : X \to X$ be mappings satisfying the following conditions:*

1. *$A(X\times X) \subseteq S(X)$;*
2. *$S(X)\}$ is a complete subspace of X;*
3. $$\begin{aligned} G(A(x,y),A(u,v),A(u,v)) &\le \alpha G(A(x,y),Su,Su)+\beta G(Sx,A(u,v),A(u,v)) \\ &\quad +\gamma G(Sx,Su,Su)+\delta G(A(u,v),Sx,Sx) \end{aligned} \tag{2}$$

$\forall x,y,u,v \in X$, $\alpha,\beta,\gamma,\delta \ge 0$ and $\alpha+2\beta+\gamma+2\delta<1$. Then A and S have a b-coupled coincidence point $(x,y)\in X\times X$, i.e., $A(x,y)=Sx$ and $A(y,x)=Sy$.
Moreover if the pair (A,S) is weakly compatible, then there exists unique $x\in X$ such that $A(x,x)=Sx$.

Proof. Consider the mapping $B : X \times X \to X$ defined by $B(x, y) = Sx$. We will check that all the hypothesis of Theorem 3.1 are satisfied. Inequality (2) of of Theorem 3.3 $\Rightarrow$ inequality (1) of Theorem 3.1 as $A(x, y) \subseteq Sx \Rightarrow A(x, y) \subseteq B(x, y)$.

Also, the weak compatibility of the pair (A, S) yields the weak compatibility of (A, B). The conditions (2) and (3) of Theorem 3.3 imply conditions (2) and (3) of Theorem 3.1 and so all the hypothesis of Theorem 3.1 are satisfied and hence by Theorem 3.1, the mappings A and S have a unique fixed point. □

Theorem 3.4. *Let (X, G) be a symmetric G-metric space and $P, S : X \to X$ be the mappings satisfying the following conditions:*

1. $P(X) \subseteq S(X)$;
2. $S(X)\}$ *is a complete subspace of* X;
3.

$$\begin{aligned} G(Px, Py, Py) \quad &\leq \quad \alpha G(Px, Sy, Sy) + \beta G(Sx, Py, Py) \\ &\quad +\gamma G(Sx, Sy, Sy) + \delta G(Py, Sx, Sx) \end{aligned} \tag{3}$$

$\forall x, y \in X, \alpha, \beta, \gamma, \delta \geq 0$ *and* $\alpha + 2\beta + \gamma + 2\delta < 1$. *Then P and S have a b-coupled coincidence point $x \in X$, i.e., $Px = Sx$ and $Py = Sy$.*

Moreover if the pair (P, S) is weakly compatible, then there exists unique $x \in X$ such that $Px = Sx$.

Proof. Consider the mappings $A, B : X \times X \to X$ defined by $A(x, y) = Px$ and $B(x, y) = Sx$. We will check that all the hypothesis of Theorem 3.1 are satisfied. Condition (1) of Theorem 3.4 $\Rightarrow$ Condition (1) of Theorem 3.1 as $P(X) \subseteq S(X) \Rightarrow A(x, y) \subseteq B(x, y)$.

Also, the weak compatibility of the pair (P, S) yields the weak compatibility of (A, S). The condition (2) and inequality (3) of Theorem 3.4 implies condition (2) and inequality (1) and so all the hypothesis of Theorem 3.1 are satisfied and hence by Theorem 3.1, the mappings P and S have a unique fixed point. □

4. Application

In this section, we study the existence of solutions of a system of nonlinear integral equations using the results proved in section 3.

Consider the following system of integral equations:

$$x(t) = \int_0^1 K_1(t, s, x(s))ds + g(t), \tag{4}$$

$$x(t) = \int_0^1 K_2(t, s, x(s))ds + g(t), \tag{5}$$

where $t \in [0, 1]$.

Let $X = C([0, 1], \mathbb{R})$ be the set of continuous functions defined on [0,1] endowed with G-metric

$$G(x, y, z) = \mid x - y \mid + \mid y - z \mid + \mid z - x \mid .$$

Then (X, G) is G-metric space. Now, we prove the existence of solution of system of integral equations (4)-(5).

Consider the following hypothesis holds:

1. $K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous.
2. For each $t, s \in [0, 1]$

$$\begin{aligned} K_1(t, s, x(s)) \quad &= \quad K_2\left(t, s, \int_0^1 K_1(s, \tau, x(\tau))d\tau + g(s)\right) \\ K_2(t, s, x(s)) \quad &= \quad K_1\left(t, s, \int_0^1 K_2(s, \tau, x(\tau))d\tau + g(s)\right) \end{aligned}$$

3. $| K_1(t,s,x) - K_1(t,s,y) | \leq \frac{1}{\kappa} | K_2(t,s,x) - K_2(t,s,y) |$, where $\kappa \geq 1$.

Then the system of integral equations (4) has a solution in X.

Proof. Define $P, S : X \to X$ by

$$\begin{aligned} Px(t) &= \int_0^1 K_1(t,s,x(s))ds + g(t) \\ Sx(t) &= \int_0^1 K_2(t,s,x(s))ds + g(t), \end{aligned}$$

where $t \in [0,1]$.
Now from (2)

$$\begin{aligned} Px(t) &= \int_0^1 K_1(t,s,x(s))ds + g(t) \\ &= \int_0^1 K_2\left(t,s,\int_0^1 K_1(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t) \\ &= \int_0^1 K_2(t,s,Px(s))ds + g(t) = SPx(t) \end{aligned}$$

$$\begin{aligned} Sx(t) &= \int_0^1 K_2(t,s,x(s))ds + g(t) \\ &= \int_0^1 K_1\left(t,s,\int_0^1 K_2(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t) \\ &= \int_0^1 K_1(t,s,Sx(s))ds + g(t) = PSx(t). \end{aligned}$$

Thus, $Px(t) = Sx(t) = PSx(t) = SPx(t)$. Therefore the pair (P,S) is weakly compatible. Also for $x, y \in [0,1]$, we have

$$\begin{aligned} G(Px,Py,Py) &= 2 \mid Px(t) - Py(t) \mid \\ &= 2 \left| \int_0^1 (K_1(t,s,x(s)) - K_1(t,s,y(s)))\, ds \right| \\ &\leq \frac{2}{\kappa}\int_a^b \mid K_2(t,s,x(s)) - K_2(t,s,y(s)) \mid ds \\ &= \frac{1}{\kappa} G(Sx,Sy,Sy), \end{aligned}$$

where $\kappa \geq 1$, $\forall x, y \in X$ and the condition (3) of Theorem 3.4 is satisfied for $\alpha = \beta = \delta = 0, \gamma = \frac{1}{\kappa}$. Thus all the conditions of Theorem 3.4 are satisfied and hence the conclusion follows. □

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