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On L_1-biharmonic hypersurfaces with constant mean curvature in the Minkowski space

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Abstract. In this paper, we study on a Riemannian manifold M^n, isometrically immersed by a map $x : M^n \to E_1^{n+1}$, in the Minkowski space $\mathbb{E}_1^{n+1}$, where the position map x satisfies the condition $L_1^2 x = 0$. This condition, as an extended version of the biharmonicity (defined by $\Delta^2 x = 0$), is called the L_1-biharmonicity condition, where L_1 stands for the linearized operator of the first variation of 2-th mean curvature of M^n in $\mathbb{E}_1^{n+1}$. A well-known conjecture of Bang-Yen Chen says that any biharmonic Euclidean submanifold has to be minimal. We discuss an analog of the Chen conjecture, replacing the Laplace operator Δ by L_1. Having assumed that M^n has at least three distinct principal curvatures and constant ordinary mean curvature, we prove that it must be 1-maximal.

1. Introduction

The role of harmonic functions and equations in physics and mathematics, applied partial differential equations, computational geometry and so on, make motivation for introducing the matter on surfaces. Sometimes, it becomes very difficult to find harmonic maps whereas biharmonic ones make help us to solve related differential equations. Since there exists no harmonic map from $\mathbb{T}^2$ into $\mathbb{S}^2$ (whatever the metrics chosen) in the homotopy class of Brower degree ± 1, it is worthwhile to find a biharmonic map $\mathbb{T}^2 \to \mathbb{S}^2$ (see in [9]). From physical points of view, biharmonic surfaces appears in applied physics, especially in elasticity and fluid mechanics ([1], [15]). Also, biharmonic maps appear in the solutions of some 4-order strongly elliptic semilinear equations and in computational geometry as the biharmonic Bezier surfaces. The variational problem associated to the bienergy functional validation on the set of Riemannian metrics for a domain resulted in the biharmonic stress-energy tensor. Obtaining proper-biharmonic maps for the study of submanifolds with certain geometric properties like as pseudo-umbilical and parallel submanifolds is practical. Bang-Yen Chen (in eighteen decade) has started to investigate the properties of biharmonic submanifolds in the Euclidean spaces. He introduced some open problems and conjectures in [7], among them, a longstanding conjecture says that every biharmonic submanifold in a Euclidean space is minimal. Chen himself has proved the conjecture for surfaces in $\mathbb{E}^3$. Later on, I. Dimitrić ([8]) has verified Chen conjecture in several different cases that satisfy the families of regular curves, submanifolds with

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constant mean curvatures, hypersurfaces with at most two distinct principal curvatures, pseudo-umbilical submanifolds of dimension $n \neq 4$ and the finite type submanifolds. T. Hasanis and T. Vlachos in [11] verified the conjecture for hypersurfaces in $\mathbb{E}^4$. Utilizing completeness, Akutagawa and Maeta ([2]) advanced the result to the global version of Chen's conjecture on biharmonic submanifolds in Euclidean spaces. Also, Chen introduced a relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen and also L.J. Alias, S.M.B. Kashani and others (see for instance, [7], [12], [16]). One can see main results in Chapter 11 of Chen's book ([6]). In [12], Kashani has introduced the notion of L_k-finite type hypersurfaces as an extension finite type ones in the Euclidean space. In [10], it is proved that only biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$ are minimal ones, and then, the result is generalized in [17] to the L_1-biharmonic hypersurfaces of $\mathbb{E}^5$.

Let M^n be an spacelike hypersurface in the pseudo-Euclidean space $\mathbb{E}_1^{n+1}$. The Laplace operator Δ stands for the linearized operator of the first variation of the mean curvature arising from normal variations of M^n in $\mathbb{E}^{n+1}$. The advanced operator L_k (where, $L_0 = \Delta$), which stands for the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations of M^n in $\mathbb{E}^{n+1}$, is defined by the explicit formula $L_k(f) = tr(P_k \circ \nabla^2 f)$ for $k = 0, 1, 2, \cdots, n-1$, and $f \in C^\infty(M)$, where P_k denotes the k-th Newton transformation associated to the second fundamental from of M and $\nabla^2 f$ is the hessian of f (see [19]). Recently, in [20], we have proved that every L_k-biharmonic spacelike hypersurface in $\mathbb{E}_1^4$ with three distinct principal curvatures is k-maximal. In this paper, we study L_1-biharmonic spacelike hypersurfaces isometrically immersed into the Lorentz-Minkowski space of arbitrary dimension, $\mathbb{E}_1^{n+1}$, having three distinct principal curvatures and constant mean curvature. Here are our main results.

Theorem 1.1. *Let $x : M^n \to \mathbb{E}_1^{n+1}$ be an isometrically immersed spacelike hypersurface satisfying the L_1-biharmonicity condition, $L_1^2 x = 0$. If M^n has constant ordinary mean curvature and non-constant 2-th mean curvature, then it has a non-constant principal curvature of multiplicity one.*

Theorem 1.2. *Every L_1-biharmonic isometrically immersed spacelike hypersurface of the Minkowski space $\mathbb{E}_1^{n+1}$ with constant mean curvature and three distinct principal curvatures is 1-maximal.*

2. Preliminaries

In this section, we recall some prerequisites from [3], [5], [10], [14], [17], [18]. By $\mathbb{E}_p^m$, we mean the Euclidean space $\mathbb{R}^m$ equipped with the scalar product $< x, y >:= -\Sigma_{i=1}^p x_i y_i + \Sigma_{j>p} x_j y_j$ (where, $0 \leq p < m$). Especially, $\mathbb{E}_0^m = \mathbb{E}^m$ and $\mathbb{E}_1^m$ are the Euclidean and Minkowski spaces of dimension m, respectively.

Let $x : M^n \to \mathbb{E}_1^{n+1}$ be an isometric immersion of a Riemannian n-dimensional manifold M into the Minkowski space $\mathbb{E}_1^{n+1}$. By the Weingarten formula we have $\bar{\nabla}_V W = \nabla_V W - < SV, W > \mathbf{N}$ for every smooth vector fields V and W on M, where, the symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M and $\mathbb{E}_1^{n+1}$ (respectively) and S is the shape operator of M associated to a timelike unit normal (local) vector field $\mathbf{N}$ on M. Since the induced metric on M^n is positive definite, the metric on M and the shape operator S can be diagonalized simultaneously, and then, we can choose a local orthonormal frame field $\{e_i\}_{1 \leq i \leq n+1}$ on M^n, where $e_1, ..., e_n$ are eigenvectors of S and $e_{n+1} = \mathbf{N}$. As usual, we denote the eigenvalues of S (the principal curvatures of M) by the functions $\lambda_1, ..., \lambda_n$ on M associated to $e_1, ..., e_n$. The elementary symmetric function is defined as $s_k := \sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1} ... \lambda_{i_k}$, so, the *k-th mean curvature* H_k of M is given by $\binom{n}{k} H_k = (-1)^k s_k$. The hypersurface M^n in $\mathbb{E}_1^{n+1}$ is called *k-maximal*, if its $(k+1)$-th mean curvature H_{k+1} is identically zero. A 0-maximal hypersurface is nothing but a maximal hypersurface in $\mathbb{E}_1^{n+1}$.

The classical Newton transformation $P_k : \chi(M) \to \chi(M)$ (for $k = 0, 1, \ldots, n$) is a linear operator on the set of vector fields on M, inductively defined by $P_0 = I$ and $P_k = \binom{n}{k} H_k I + S \circ P_{k-1}$ for $k = 1, \ldots, n$ where I denotes the identity transformation on $\chi(M)$. An explicit expression of Newton transformation as $P_k = \sum_{j=0}^{k} \binom{n}{k-j} H_{k-j} S^j$, for $k = 0, 1, \ldots, n$ gives $P_n = 0$ (by using the Cayley-Hamilton theorem which says that any operator is annihilated by its characteristic polynomial). According to definition of P_k, it is self-adjoint and commute with S, whose corresponding eigenvalues are given by

$$\mu_{i,k}(p) = (-1)^k \sum_{1\leq i_1<\cdots<i_k\leq n, i_j\neq i} \lambda_{i_1}(p)\cdots\lambda_{i_k}(p), \tag{2.1}$$

for $i = 1, \cdots, n$ and $k = 1, \cdots, n$. We recall some useful formulae on Newton transformations from [5], [19].

$$\begin{aligned} &tr(P_k) = c_k H_k, \\ &tr(S \circ P_k) = -c_k H_{k+1}, \\ &tr(S^2 \circ P_k) = \binom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}), \end{aligned} \tag{2.2}$$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$. Also, we recall the linearized operator $L_k : C^\infty(M) \to C^\infty(M)$ defined by

$$L_k(f) = tr(P_k \circ \nabla^2 f), \tag{2.3}$$

where, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f defined by $< \nabla^2 f(X), Y > := < \nabla_X(\nabla f), Y >$, where $X, Y \in \chi(M)$, ∇f is the gradient of f. Based on the local orthonormal frame $\{e_1, \ldots, e_n\}$, $L_k(f)$ is given by

$$L_k(f) = \sum_{i=1}^{n} \mu_{i,k}(e_ie_if - \nabla_{e_i}e_if). \tag{2.4}$$

From now on, we concentrate on connected orientable isometrically immersed spacelike hypersurface in the Minkowski space, $x : M^n \to \mathbb{E}^{n+1}_1$, having three distinct principal curvatures and constant ordinary mean curvature H. By definition, M^n is said to be L_1-biharmonic, if its position vector field satisfies the condition $L_1^2 x = 0$. By the equality $L_k x = c_k H_{k+1} N$ from [19], [13], the condition $L_1^2 x = 0$ has another equivalent expression as $L_1(H_2 N) = 0$. Clearly, every 1-maximal hypersurface is L_1-biharmonic. By formulae in [4], [13], [19], for every integer k (where $0 \leq k \leq n-1$) we have

$$L_1^2 x = -2\binom{n}{2}[2P_2 - 3\binom{n}{2}H_2 I]\nabla H_2 + 2\binom{n}{2}[L_1 H_2 - \binom{n}{2}H_2(nHH_2 + (n-2)H_3)]\mathbf{N}. \tag{2.5}$$

Hence, identifying the normal and tangent parts of (2.5), one obtains necessary and sufficient conditions for M^n to be L_1-biharmonic in $\mathbb{E}^{n+1}_1$, namely

$$(i)\ L_1 H_2 = \binom{n}{2}H_2(nHH_2 + (n-2)H_3), \quad (ii)\ P_2\nabla H_2 = \frac{3}{2}\binom{n}{2}H_2\nabla H_2. \tag{2.6}$$

3. Results

In order to prove Theorems 1.1 and 1.2, we state the following two auxiliary lemmas.

3.1. Auxiliary Lemmas

Lemma 3.1. *Let $x : M^n \to \mathbb{E}^{n+1}_1$ be an L_1-biharmonic spacelike hypersurface in the Minkowski $(n+1)$-space with three distinct principal curvatures, constant mean curvature and non-constant 2-th mean curvature. Then, with respect to orthonormal (local) tangent frame $\{e_1, \ldots, e_n\}$ of the principal directions on M^n, we have*

(i) $\nabla_{e_1}e_1 = 0$,

(ii) $\nabla_{e_i}e_1 = \alpha e_i$, *for* $i = 2, \ldots, n-1$, *where,* $\alpha := \frac{e_1(\lambda)}{\lambda_1 - \lambda}$,

(iii) $\nabla_{e_n}e_1 = -\beta e_n$, *where* $\beta = \frac{e_1(\lambda_1 + (n-2)\lambda)}{(\lambda_1 - \eta)}$,

(iv) $\nabla_{e_i}e_i = -\alpha e_1 + \sum_{k=2, k\neq i}^{n-1} \omega_{ii}^k e_k + \frac{e_n(\lambda)}{nH + \lambda_1 + (n-1)\lambda}e_n$, *for* $i = 2, 3, \ldots, n-1$,

(v) $\nabla_{e_i} e_j = \sum_{k=2}^{n-1} \omega_{ij}^k e_k$, *for* $i, j = 2, 3, \ldots, n-1$, *where* $i \neq j$,
(vi) $\nabla_{e_1} e_n = 0$,
(vii) $\nabla_{e_n} e_n = \beta e_1$,
(viii) $\nabla_{e_i} e_n = \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} e_i$ *for* $i = 2, 3, \ldots, n-1$,
where $\omega_{ki}^i = 0$ *and* $\omega_{ki}^j + \omega_{kj}^i = 0$, *for* $i, j, k = 1, \cdots, n$.

Proof. By assumption, there exists an open connected subset U of M, on which we have $\nabla H_2 \neq 0$. By (2.6(i)), $e_1 := \frac{\nabla H_2}{\|\nabla H_2\|}$ is an eigenvector of P_2 with the corresponding eigenvalue $\frac{3}{4}n(n-1)H_2$ on U. Without loss of generality, we can assume that U = M and take a suitable orthonormal (local) basis $\{e_1, \cdots, e_n\}$ for the tangent bundle of M^n, consisting of the eigenvectors the shape operator S such that $Se_i = \lambda_i e_i$ and $P_2 e_i = \mu_{i,2} e_i$, (for $i = 1, \cdots, n$). Hence, $\mu_{1,2} = \dfrac{3}{4}n(n-1)H_2$. We use the notation $\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k$ for $i, j = 1, \ldots, n$. By the compatibility conditions $\nabla_{e_k} < e_i, e_i >= 0$ and $\nabla_{e_k} < e_i, e_j >= 0$, we have identities

$$(i)\ \omega_{ki}^i = 0, \quad (ii)\ \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.1}$$

for $i, j, k = 1, \cdots, n$. Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.2}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{3.3}$$

Using the equality $\mu_{1,2} = \dfrac{3}{4}n(n-1)H_2$ and the definition of H_2,

$$H_2 = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j, \tag{3.4}$$

we get

$$H_2 = \frac{4}{n(n-1)} \lambda_1(\lambda_1 + nH), \tag{3.5}$$

and by differentiating along e_1, we obtain

$$e_1(\lambda_1) \neq 0, \quad e_i(\lambda_1) = 0 \quad i = 2, \ldots, n. \tag{3.6}$$

Using the decomposition $\nabla H_2 = \sum_{i=1}^n e_i(H_2) e_i$, by assumption $e_1 = \frac{\nabla H_2}{\|\nabla H_2\|}$, we have

$$e_1(H_2) \neq 0, \qquad e_i(H_2) = 0 \quad i = 2, \ldots, n. \tag{3.7}$$

One can compute that

$$[e_i, e_j](\lambda_1) = 0, \quad i, j = 2, \ldots, n,$$

which yields directly

$$\omega_{ij}^1 = \omega_{ji}^1, \tag{3.8}$$

for $i \neq j$ and $i, j = 2, \ldots, n$. Since M^n has three distinct principal curvatures, we can assume that $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n \neq \lambda$, hence $\lambda_n = nH - \lambda_1 - (n-2)\lambda$.

Now, for part (i), it is enough to show that $\omega_{11}^k = 0$ for $k = 1, \ldots, n$. Taking $j = 1$ and $i = 2, \ldots, n$ in the equality (3.2) and using (3.6), we obtain $\omega_{12}^1 = \omega_{13}^1 = \cdots = \omega_{1n}^1 = 0$, which gives, by (3.1(ii)), the result $\omega_{11}^k = 0$ for $k = 2, \ldots, n$. So, it remains to see that $\omega_{11}^1 = 0$, which is given in (3.1(i)).

For part (ii), it must to be proved that

$$\omega_{i1}^k = \{^{\alpha,\ (k=i)}_{0,\ (k\neq i)}$$

for $i = 2, \cdots, n-1$ and $k = 1, 2, \cdots, n$. For the cases $k = i = 2, \cdots, n-1$, we use the equality (3.2) (by interchanging i, j in (3.2) and then taking $j = 1$) to get $e_1(\lambda) = (\lambda_1 - \lambda)\omega_{i1}^i$, which gives $\omega_{i1}^i = \alpha$. For other cases, fix an arbitrary integer i where $2 \leq i \leq n-1$. we try to show $\omega_{i1}^k = 0$ for every integer k where $1 \leq k \leq n$ and $k \neq i$. For $k = 1$ we have $\omega_{i1}^1 = 0$ by (3.1(i)). For other cases, we take $k = 1$ and $2 \leq j \neq i \leq n-1$ in (3.3) to get $\omega_{i1}^j = 0$ for $j = 2, \cdots, n-1$. Finally, from (3.2) we get $\omega_{i1}^n = 0$. So, the proof of part (ii) is complete.

For part (iii), we must show that $\omega_{n1}^k = 0$ for $k = 1, \cdots, n-1$, and $\omega_{n1}^n = -\beta$. For the case $k = 1$, by taking $i = n$ and $k = 1$ in (3.1(i)), we have $\omega_{n1}^1 = 0$. For $k = 2, \cdots, n-1$, remember that in part (i) we had $\omega_{k1}^n = 0$ for $k = 2, \cdots, n-1$ which, by (3.1(ii)) and (3.8), gives $\omega_{n1}^k = \omega_{nk}^1 = \omega_{kn}^1 = 0$ for $k = 2, \cdots, n-1$. Finally, for $k = n$, by putting $i = 1$ and $j = n$ in (3.2), we obtain $\omega_{n1}^n = -\beta$.

For part (iv), we must show that $\omega_{ii}^1 = -\alpha$ and $\omega_{ii}^i = 0$ for $i = 2, \cdots, n-1$, and also $\omega_{ii}^n = \frac{-e_n(\lambda}{\lambda_n}$. For first one, from part (ii) we have $\omega_{i1}^i = \alpha$ which, by (3.1(ii)), gives $\omega_{ii}^1 = -\alpha$. Also, (3.1(ii)) gives $\omega_{ii}^i = 0$ for $i = 2, \cdots, n-1$. Finally, for the last result, from (3.2) we have $\omega_{in}^i = \frac{e_n(\lambda}{\lambda_n-\lambda}$, for $i = 2, \cdots, n-1$, which gives (by (3.1(ii))) the result. Hence, the proof of part (ii) is complete.

For part (v), we have to show that $\omega_{ij}^k = 0$ for $k = 1, n$. It is enough to use (3.2) for special values of k (i.e. $k = 1, n$) to get $\omega_{i1}^j = \omega_{in}^j = 0$ for $i, j = 2, \ldots, n-1$ where $i \neq j$, which gives the result, by (3.8).

For part (vi), we show that $\omega_{1n}^k = 0$ for $k = 1, \cdots, n$. In the case $k = 1, n$, clearly, using (3.1(i))and (3.8) we get the result. In remained cases that $k = 2, \cdots, n-1$, from (3.3) we have $(\lambda_n - \lambda)\omega_{1n}^k = (\lambda_1 - \lambda)\omega_{n1}^k$, which by the final statement in the proof of part (iii) (i.e. $\omega_{n1}^k = 0$) gives $\omega_{n1}^k = 0$ for $k = 2, \cdots, n-1$. Hence, the proof of part (vi) is complete.

In part (vii), we have to show that

$$\omega_{nn}^k = \{^{\beta,\ (k=1)}_{0,\ (k=2,\cdots,n).}$$

For the case $k = 1$, we put $i = 1$ and $j = n$ in the equality (3.2) to get $e_1(\lambda_n) = (\lambda_1 - \lambda_n)\omega_{n1}^n$ which gives, by (3.1(ii)), $\omega_{nn}^1 = -\omega_{n1}^n = \beta$. Now, for $k = 2, \cdots, n-1$, first, from (3.2), by taking $i = k$, $j = 2, \ldots, n-1$ where $j \neq k$, we get $e_k(\lambda) = 0$ which together with $e_k(\lambda_1) = 0$ (from (3.6)) gives $e_k(\lambda_n) = 0$ for $k = 2, \cdots, n-1$. Hence, by (3.2) again, we obtain $\omega_{nk}^n = 0$ and then (by (3.1(ii))) we have $\omega_{nn}^k = 0$ for $k = 2, \cdots, n-1$. Finally, the result $\omega_{nn}^n = 0$ is given in (3.1(i)).

In the rest part (i.e. (viii)), we will see that

$$\omega_{in}^k = \{^{\frac{e_n(\lambda)}{\lambda_n},\ (k=i)}_{0,\ (k\neq i,\ k=1,\cdots,n)}$$

for $i = 2, \cdots, n-1$. Fix an arbitrary integer i where $2 \leq i \leq n-1$. Now, for $k = 1$, the claim $\omega_{in}^1 = 0$ is a direct consequence of the result $\omega_{i1}^n = 0$ in part (ii). For $k = 2, \cdots, n-1$ where $k \neq i$, from part (v) we have $\omega_{ik}^n = 0$ which, by (3.1(ii)), gives $\omega_{in}^k = 0$. For $k = i$, from part (iv) we have $\omega_{ii}^n = -\frac{e_n(\lambda)}{\lambda_n}$ which, by (3.1(ii)), gives $\omega_{in}^i = \frac{e_n(\lambda)}{\lambda_n}$. For $k = n$, it is a direct consequence of (3.1(ii)) that $\omega_{in}^n = 0$. So, the proof of part (viii) is complete. □

Lemma 3.2. *Let M^n be an L_1-biharmonic spacelike hypersurface in Lorentz-Minkowski space $\mathbb{E}_1^{n+1}$, with three distinct principal curvatures, constant ordinary mean curvature and non-constant 2-th mean curvature. Then, there exists a locally moving orthonormal tangent frame $\{e_1, \cdots, e_n\}$ of principal directions on M^n with associated principal curvatures $\lambda_1, \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \lambda_n = \eta$, satisfying $e_n(\lambda) = 0$ and*

$$e_1(\lambda)e_1(\lambda_1 + (n-2)\lambda) = \lambda\eta(\lambda_1 - \lambda)(\lambda_1 - \eta). \tag{3.9}$$

Proof. For convenience we use the notations $\alpha := \frac{e_1(\lambda)}{\lambda_1-\lambda}$ and $\beta := \frac{e_1(\lambda_1+(n-2)\lambda)}{(\lambda_1-\eta)}$. By computing both of sides of the Gauss curvature tensor formula

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{3.10}$$

by means of Lemma 3.1 and the well-known Gauss equation (see [18], ch.4, Theorem 5), we can obtain different equalities on the moving tangent frame $\{e_1, \cdots, e_n\}$ that was introduced in Lemma 3.1. We consider three different cases as follow.
Case 1: Take $X := e_1$, $Y := e_2$ in (3.10). Putting $Z := e_1$ and $Z := e_n$, respectively, we obtain two following equalities

$$e_1(\alpha) + \alpha^2 = -\lambda_1\lambda, \tag{3.11}$$

$$e_1\left(\frac{e_n(\lambda)}{\eta+\lambda}\right) + \alpha\frac{e_n(\lambda)}{\eta+\lambda} = 0. \tag{3.12}$$

Case 2: Take $Z := e_1$, $Y := e_n$ in (3.10). Putting $X := e_1$ and $X := e_3$, respectively, we get the following equalities

$$-e_1(\beta) + \beta^2 = -\lambda_1\eta, \tag{3.13}$$

$$e_n(\alpha) + (\alpha+\beta)\frac{e_n(\lambda)}{\eta+\lambda} = 0. \tag{3.14}$$

Case 3: Putting $X := Z := e_n$, $Y := e_2$ in (3.10), we get

$$-e_n\left(\frac{e_n(\lambda)}{\eta+\lambda}\right) + \alpha\beta - \left(\frac{e_4(\lambda)}{\eta+\lambda}\right)^2 = \lambda\eta. \tag{3.15}$$

Now, from condition (2.6), using (2.4) and Lemma (3.1), we get

$$\begin{aligned}(\lambda_1 + nH)e_1e_1(H_2) + \left[\frac{¢}{¢}(n-2)(\lambda+nH)\alpha + (\lambda_1 + (n-2)\lambda)\beta\right]e_1(H_2)& \\ - \frac{n(n-1)(n-2)}{2}H_2(2HH_2 - H_3) = 0.&\end{aligned} \tag{3.16}$$

On the other hand, using equation (3.7) in the proof of Lemma (3.1), we have

$$e_ie_1(H_2) = 0, \qquad i = 2, \ldots, n. \tag{3.17}$$

Now, differentiating α and β along e_n, (using (3.6)) we get two equalities as follow

$$(\lambda_1 - \lambda)e_n(\alpha) - \alpha e_n(\lambda) = e_ne_1(\lambda),$$

$$(\lambda_1 - \eta)e_n(\beta) + (n-2)\beta e_n(\lambda) = (n-2)e_ne_1(\lambda),$$

which, comparing with each other (and eliminating $e_ne_1(\lambda)$), give

$$(\lambda_1 - \eta)e_n(\beta) = (n-2)\left[(\lambda_1-\lambda)e_n(\alpha) - (\alpha+\beta)e_n(\lambda)\right].$$

Putting the value of $e_n(\alpha)$ from (3.14) in the above equation, we find

$$e_n(\beta) = \frac{e_n(\lambda)(n-2)(\alpha+\beta)(n\lambda - nH)}{(\lambda_1-\eta)\eta}.$$

Differentiating (3.16) along e_n, using (3.17) and (3.14) and substituting the value $e_n(\beta)$ in the result, we get

$$(n-2)e_n(\lambda)\left[\frac{(\alpha+\beta)A}{\lambda_1-\eta}e_1(H_2) - H_2\eta B\right] = 0. \tag{3.18}$$

where $A := 4nH\lambda_1 - 2\lambda_1{}^2 - 2(n-1)\lambda\lambda_1 + 2n(n-1)H\lambda - 2n^2H^2$ and
$B := n^2H^2 + 3\lambda_1^2 + (3(n-2)^2 - 3)\lambda^2 + (2n - 4n(n-2))H\lambda - 4nH\lambda_1 + 6(n-2)\lambda\lambda_1$.
Now we claim that $e_n(\lambda) = 0$. If not, Having assumed $e_n(\lambda) \neq 0$, we have

$$\frac{(\alpha+\beta)A}{\lambda_1 - \eta} e_1(H_2) - H_2\eta B = 0, \tag{3.19}$$

By differentiating (3.19) along e_n, we have

$$\frac{(\alpha+\beta)\left[A((n-4)\lambda_1 + 2(n-2)^2\lambda + (n - 2n(n-2))H) + C\right]e_1(H_2)}{(\lambda_1-\eta)^2} + H_2D = 0, \tag{3.20}$$

where $C := -(2n(n-1)H + 2(n-1)\lambda_1)\eta(\lambda_1 - \eta)$ and
$D =: -\eta^2[(6(n-2)^2 - 6)\lambda + (2n - 4n(n-2))H + 6(n-2)\lambda_1] + (n-1)\eta B$.
Finally, from (3.19) and (3.20) (by eliminating $e_1(H_2)$), we obtain

$$-AD(\lambda_1 - \eta) = \eta B[A((n-4)\lambda_1 + 2(n-2)^2\lambda + (n - 2n(n-2))H) + C] \tag{3.21}$$

After four times differentiating (3.21) along e_n, we get that $nH = \lambda_1$, which is impossible since H is assumed to constant but λ_1 is non-constant. Consequently, our claim is proved (i.e. $e_n(\lambda) = 0$). Therefore, (3.15) reduces to (3.9) □

3.2. Main results

Now, we prove main theorems.

Proof of Theorem 1.1 . By assumption, there exists an open connected subset U of M, on which we have $\nabla H_2 \neq 0$. We take $e_1 := \frac{\nabla H_2}{\|\nabla H_2\|}$, which is an eigenvector of P_2 with eigenvalue $\frac{3}{4}n(n-1)H_2$ on U, by the equation (2.6)(i). Without loss of generality, we assume that $U = M$ and then, we choose suitable principal directions $e_2, \cdots, e_n$ on M (other than e_1) such that $\{e_1, \cdots, e_n\}$ be an orthonormal tangent bundle on M. We denote the principal curvatures of M by $\lambda_1, \cdots, \lambda_n$ according to $e_1, \cdots, e_n$, respectively. Therefore, we have $Se_i = \lambda_i e_i$ and then, by the equation 2.1, $P_2 e_i = \mu_{i,2} e_i$, (for $i = 1, \cdots, n$), and in special case, we note that $\mu_{1,2} = \dfrac{3}{4}n(n-1)H_2$. Clearly, equations (3.1)- (3.7) can be verified here. Using the notation $\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k$ for $i, j = 1, \ldots, n$, from the compatibility conditions $\nabla_{e_k} < e_i, e_i >= 0$ and $\nabla_{e_k} < e_i, e_j >= 0$, we get

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.22}$$

for $i, j, k = 1, \cdots, n$, where $i \neq j$. Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.23}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{3.24}$$

Having assumed $\lambda_j = \lambda_1$ for some integer $j \neq 1$. Taking $i = 1$, from (3.23) we obtain

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts the first expression of (3.6). So, the main claim of lemma is verified. □

Proof of Theorem 1.2. By differentiating (3.5) along e_1, and using the definition of β in Lemma 3.2, we have

$$e_1(H_2) = -\frac{4(n-2)}{n(n-1)}(2\lambda_1 + nH)e_1(\lambda) + \frac{4}{n(n-1)}(\lambda_1 - \eta)(2\lambda_1 + nH)\beta, \tag{3.25}$$

which by differentiating (3.5) along e_1 and using lemma (3.2) and equations (3.13) and (3.11) of Lemma 3.1, we obtain

$$e_1e_1(H_2) = P_{0,4} + \left[-n\alpha + 3\beta + 2\frac{(\lambda_1-\eta)\beta - (n-2)(\lambda_1-\lambda)\alpha}{2\lambda_1 + nH}\right]e_1(H_2), \tag{3.26}$$

where,

$$P_{0,4} := \frac{4(n-2)}{n(n-1)}\lambda_1\lambda(\lambda_1-\lambda)(2\lambda_1+nH) + \frac{4}{n(n-1)}\eta(2\lambda_1+nH)((3n-2)\lambda_1\lambda + 2{\lambda_1}^2 + 2nH\lambda + nH\lambda_1).$$

Combining (3.16) with (3.26) gives

$$(P_{1,2}\alpha + P_{2,2}\beta)e_1(H_2) = P_{3,6}, \tag{3.27}$$

where

$$P_{1,2} := (\frac{2(n-2)\lambda_1}{2\lambda_1+nH} - 3n + 4)(\lambda_1-\lambda) - 2(\lambda+nH),$$
$$P_{2,2} := \frac{-2\lambda_1(\lambda_1-\eta)}{2\lambda_1+nH} - 2\eta + (n-2)\lambda + 3(2\lambda_1+nH)$$

and

$$P_{3,6} := \frac{-4(\lambda_1+nH)(2\lambda_1+nH)\lambda\lambda_1}{n(n-1)}[(n-2)(\lambda_1-\lambda) + (3n-2)\eta + 2nH\lambda\eta + \lambda_1\eta(2\lambda_1+nH)]$$
$$-\frac{1}{2}n(n-1)(n-2)H_2(2HH_2 - H_3)$$

are polynomials in terms of λ_1,λ and η of degrees 2, 2 and 6, respectively.

Differentiating (3.27) along e_1 and using (3.13), (3.11) and (3.27), we get following relation

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}e_1(H_2), \tag{3.28}$$

where $P_{4,8}$, $P_{5,8}$ and $P_{6,5}$ are polynomials in terms of λ and λ_1 of degrees 8, 8 and 5 respectively. Also, we have

$$e_1(H_2) = \frac{4}{n(n-1)}(2\lambda_1+nH)\left(\beta(\lambda_1-\eta) - (n-2)\alpha(\lambda_1-\lambda)\right). \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$\begin{aligned}&\left(P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1+nH)\right)\alpha \\ &+\left(P_{5,8} - \frac{4}{n(n-1)}P_{6,5}(\lambda_1-\eta)(2\lambda_1+nH)\right)\beta = 0.\end{aligned} \tag{3.30}$$

On the other hand, combining (3.29) with (3.27), we find

$$P_{2,2}(\lambda_1-\eta)(2\lambda_1+nH)\beta^2 - P_{1,2}(n-2)(\lambda_1-\lambda)(2\lambda_1+nH)\alpha^2 = \Phi, \tag{3.31}$$

where Φ is given by

$$\Phi := \lambda\eta(2\lambda_1+nH)\left(\frac{¢}{¢}P_{2,2}(n-2)(\lambda_1-\lambda) - P_{1,2}(\lambda_1-\eta)\right) + \frac{n(n-1)}{4}P_{3,6}. \tag{3.32}$$

Using (3.30) and (3.31), we get

$$\alpha^2 = \frac{\frac{4}{n(n-1)}P_{6,5}(\lambda_1-\eta)(2\lambda_1+nH) - P_{5,8}}{P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1+nH)}\lambda\eta, \quad \beta^2 = \frac{-\frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1+nH) - P_{4,8}}{P_{5,8} - \frac{4}{n(n-1)}P_{6,5}(\lambda_1-\eta)(2\lambda_1+nH)}\lambda\eta$$

.

Eliminating α^2 and β^2 from (3.31), we obtain

$$\lambda\eta(2\lambda_1+nH)\left[P_{1,2}(n-2)(\lambda_1-\lambda)(P_{5,8}-\tfrac{4}{n(n-1)}P_{6,5}(\lambda_1-\eta)(2\lambda_1+nH))^2\right]$$
$$-\lambda\eta(2\lambda_1+nH)\left[P_{2,2}(\lambda_1-\eta)(P_{4,8}+\tfrac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1+nH))^2\right]$$
$$=\Phi(P_{5,8}-\frac{4}{n(n-1)}P_{6,5}(\lambda_1-\eta)(2\lambda_1+nH))(P_{4,8}+\frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1+nH)), \tag{3.33}$$

which is a polynomial equation of degree 22 in terms of λ and λ_1.

Now consider an integral curve of e_1 passing through $p=\gamma(t_0)$ as $\gamma(t)$, $t\in I$. Since $e_i(\lambda_1)=e_i(\lambda)=0$ for $i=2,\dots,n$ and $e_1(\lambda_1)$, $e_1(\lambda)\neq 0$, we can assume $t=t(\lambda)$ and $\lambda_1=\lambda_1(\lambda)$ in some neighborhood of $\lambda_0=\lambda(t_0)$. Using (3.30), we have

$$\frac{d\lambda_1}{d\lambda}=\frac{d\lambda_1}{dt}\frac{dt}{d\lambda}=\frac{e_1(\lambda_1)}{e_1(\lambda)}=\frac{(\lambda_1-\eta)\beta-(n-2)(\lambda_1-\lambda)\alpha}{(\lambda_1-\lambda)\alpha}$$
$$=\frac{\left(P_{4,8}+\frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1-\lambda)(2\lambda_1+nH)\right)(\lambda_1-\eta)}{\left(\frac{4}{n(n-1)}P_{6,5}(\lambda_1-\eta)(2\lambda_1+nH)-P_{5,8}\right)(\lambda_1-\lambda)}-(n-2) \tag{3.34}$$

Differentiating (3.33) with respect to λ and substituting $\frac{d\lambda_1}{d\lambda}$ from (3.34), we get

$$f(\lambda_1,\lambda)=0, \tag{3.35}$$

another algebraic equation of degree 30 in terms of λ_1 and λ.

We rewrite (3.33) and (3.35) respectively in the following forms

$$\sum_{i=0}^{22} f_i(\lambda_1)\lambda^i, \qquad \sum_{i=0}^{30} g_i(\lambda_1)\lambda^i, \tag{3.36}$$

where $f_i(\lambda_1)$ and $g_j(\lambda_1)$ are polynomial functions of λ_1. We eliminate λ^{30} between these two polynomials of (3.36) by multiplying $g_{30}\lambda^8$ and f_{22} respectively on the first and second equations of (3.36), we obtain a new polynomial equation in λ of degree 29. Combining this equation with the first equation of (3.36), we successively obtain a polynomial equation in λ of degree 28. In a similar way, by using the first equation of (3.36) and its consequences we are able to gradually eliminate λ. At last, we obtain a non-trivial algebraic polynomial equation in λ_1 with constant coefficients. Therefore, we conclude that the real function λ_1 must be a constant, which is a contradiction. Hence H_2 is constant on M^n. If $H_2\neq 0$, by using (2.6) we obtain that H_3 is constant. Therefore all the mean curvatures H_r are constant functions, this is equivalent to M^n is isoparametric. An isoparametric spacelike hypersurface of Lorentz-Minkowski space can have at most two distinct principal curvatures ([5]), which is a contradiction. So $H_2=0$.

□

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