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Li-Yorke Chaotic Properties of Abstract Differential Equations of First Order

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Abstract. In this paper, we analyze the Li-Yorke chaotic properties of abstract non-degenerate differential equations of first order in Banach and Fréchet spaces. We investigate the property of Li-Yorke chaos for sequences of linear (not necessarily continuous) operators. Li-Yorke chaotic properties of translation semigroups and strongly continuous semigroups induced by semiflows are studied, as well.

1. Introduction and preliminaries

The notion of Li-Yorke chaos was introduced by Li and Yorke [17] in 1975, and it has attracted great attention after that. Li-Yorke chaotic linear continuous operators on Banach and Fréchet spaces have been thoroughly analyzed in [5] and [7] (cf. also [1], [8] and [13]). It is said that a linear continuous operator T acting on a Fréchet space X is Li-Yorke chaotic iff there exists an uncountable set $S \subseteq X$ (scrambled set) such that for each pair $x,\ y \in S$ of distinct points we have that

$$\liminf_{k\to+\infty} d\Big(T^k x, T^k y\Big) = 0 \text{ and } \limsup_{k\to+\infty} d\Big(T^k x, T^k y\Big) > 0,$$

where $d(\cdot,\cdot)$ is the metric on X defined by the equation (1) below. If S can be chosen to be dense in X, then we say that T is densely Li-Yorke chaotic.

The main aim of this paper is to transfer the results of Bernardes Jr et al. [7] to strongly continuous semigroups in Fréchet spaces, as well as to continue the researches of Wu [19] and Conejero et al. [9]. The organization of paper can be briefly described as follows. In the second section of paper, we examine the possibility to extend results presented in [7] to sequences of linear, in general, non-continuous operators. In the third section, we introduce the notion of a $\tilde{X}$-Li-Yorke (semi)-irregular vector of a strongly continuous semigroup $(T(t))_{t\geq 0}$. We prove in Theorem 3.7 that, under some conditions, any neighborhood of a $\tilde{X}$-Li-Yorke semi-irregular vector x of $(T(t))_{t\geq 0}$ contains a $\tilde{X}$-Li-Yorke irregular vector of $(T(t))_{t\geq 0}$. From the point of view of possible applications, the most important results of third section are Theorem 3.13 and Corollary 3.14. In the fourth section of paper, we enquire into the basic Li-Yorke chaotic properties of

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translation semigroups and strongly continuous semigroups induced by semiflows, acting on various classes of weighted function spaces.

Throughout this paper, we assume that X is an infinite-dimensional Fréchet space over the field od complex numbers, and that the topology of X is induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms. The translation invariant metric $d : X \times X \to [0,\infty)$ is defined by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)},\ x,\ y \in X. \tag{1}$$

Let us recall that $d(\cdot,\cdot)$ has the following properties:

$$d(x+u, y+v) \leq d(x,y) + d(u,v), \quad x,\ y,\ u,\ v \in X, \tag{2}$$

$$d(cx, cy) \leq (|c|+1)d(x,y),\ c \in \mathbb{C},\ x,\ y \in X, \tag{3}$$

and

$$d(\alpha x, \beta x) \geq \frac{|\alpha-\beta|}{1+|\alpha-\beta|} d(0,x), \quad x \in X,\ \alpha,\ \beta \in \mathbb{C}. \tag{4}$$

If $(Z, \|\cdot\|_Z)$ is a Banach space under consideration, then it will be assumed that the metric on Z is given by $d_Z(x,y) := \|x-y\|_Z,\ x,\ y \in Z$; the norm on X will be abbreviated to $\|\cdot\|$. By I we denote the identity operator on X. Suppose now that Y is another Fréchet space over the field of complex numbers and the topology of Y is induced by the fundamental system $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms. By $d_Y(\cdot,\cdot)$ we denote the induced metric on Y (cf. (1)), and by $L(X,Y)$ we denote the space which consists of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. Throughout the paper, $\tilde{X}$ denotes a closed linear subspace of X. Let $\mathcal{B}$ be the family of bounded subsets of X and let $p_{n,B}(T) := \sup_{x\in B} p_n^Y(Tx)$, $n \in \mathbb{N}$, $B \in \mathcal{B}$, $T \in L(X,Y)$. Then $p_{n,B}(\cdot)$ is a seminorm on $L(X,Y)$ and the calibration $(p_{n,B})_{(n,B)\in\mathbb{N}\times\mathcal{B}}$ induces a Hausdorff locally convex topology on $L(X,Y)$. In the case that $(X,\|\cdot\|)$ or $(Y,\|\cdot\|_Y)$ is a Banach space, then the distance of two elements $x,\ y \in X$ $(x,\ y \in Y)$ will be defined by $d(x,y) := \|x-y\|$ $(d_Y(x,y) := \|x-y\|_Y)$. By A we denote a closed linear operator acting on X; unlesss stated otherwise, $C \in L(X)$ denotes an injective operator satisfying $CA \subseteq AC$. The domain, range, point spectrum and adjoint operator of A are denoted by $D(A)$, $\rho(A)$, $R(A)$, $\sigma_p(A)$ and A^*, respectively. Since no confusion seems likely, we will identify A with its graph. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda-A)^{-1}C \in L(X)\}$. Suppose now that F is a linear subspace of X. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. If $\tilde{X}$ is a closed linear subspace of X, then $\tilde{X}$ is a Fréchet space itself and the fundamental system of seminorms which induces the topology on $\tilde{X}$ is $(p_{n|\tilde{X}})_{n\in\mathbb{N}}$.

2. Li-Yorke chaos for single operators

Throughout this section, we assume that X and Y are two infinite-dimensional Fréchet spaces over the field of complex numbers and that the topology of Y is induced by the fundamental system $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms. Our main aim is to investigate the basic Li-Yorke chaotic properties of a sequence $(T_k)_{k\in\mathbb{N}}$ of linear, not necessarily continuous, mappings between the spaces X and Y.

Definition 2.1. *Suppose that, for every $k \in \mathbb{N}$, $T_k : D(T_k) \to Y$ is a linear operator and $\tilde{X}$ is a closed linear subspace of X. Then we say that the sequence $(T_k)_{k\in\mathbb{N}}$ is $\tilde{X}$-Li-Yorke chaotic iff there exists an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ such that for each pair $x,\ y \in S$ of distinct points we have that*

$$\liminf_{k\to+\infty} d_Y\Big(T_k x, T_k y\Big) = 0 \text{ and } \limsup_{k\to+\infty} d_Y\Big(T_k x, T_k y\Big) > 0; \tag{5}$$

the sequence $(T_k)_{k\in\mathbb{N}}$ is said to be densely $\tilde{X}$-Li-Yorke chaotic iff S can be chosen to be dense in $\tilde{X}$. A linear operator $T : D(T) \to Y$ is said to be (densely) $\tilde{X}$-Li-Yorke chaotic iff the sequence $(T_k \equiv T^k)_{k\in\mathbb{N}}$ is. If $\tilde{X} = X$, then we also say *that the sequence* $(T_k)_{k\in\mathbb{N}}$ *(the operator* T*) is Li-Yorke chaotic. The set* S *is said to be* $\tilde{X}$*-scrambled set (scrambled set, in the case that* $\tilde{X} = X$*) of the sequence* $(T_k)_{k\in\mathbb{N}}$ *(the operator* T*). If (5) holds, then we call* (x, y) *a Li-Yorke pair for* $(T_k)_{k\in\mathbb{N}}$*.*

Besides of above, we can also introduce some other notions of (subspace) Li-Yorke chaoticity of sequence $(T_k)_{k\in\mathbb{N}}$, like generic Li-Yorke chaoticity, densely weak Li-Yorke chaoticity, generically weak Li-Yorke chaoticity, Li-Yorke sensitivity and spatio-temporal chaoticity (cf. [7, p. 1724] and [19, Definition 1.1]).

Now we will introduce the notions of $\tilde{X}$-Li-Yorke irregular vectors for $(T_k)_{k\in\mathbb{N}}$ and $\tilde{X}$-Li-Yorke semi-irregular vectors for $(T_k)_{k\in\mathbb{N}}$ following the approaches of Beauzamy [4] (Banach space setting) and Bernardes Jr et al. [7] (Fréchet space setting):

Definition 2.2. *Let* $\tilde{X}$ *be a closed linear subspace of* X*, let for each* $k \in \mathbb{N}$ $T_k : D(T_k) \to Y$ *be a linear operator, and let* $x \in \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$*. Then it is said that* x *is:*

(i) a $\tilde{X}$*-Li-Yorke irregular vector for* $(T_k)_{k\in\mathbb{N}}$ *iff there exists a sequence* (k_n) *of non-negative integers such that the set* $\{T_k x : k \in \mathbb{N}\}$ *is unbounded in* Y *and* $\lim_{n\to+\infty} T_{k_n}x = 0$*;*

(ii) a $\tilde{X}$*-Li-Yorke semi-irregular vector for* $(T_k)_{k\in\mathbb{N}}$ *iff there exists a sequence* (k_n) *of non-negative integers such that* $T_k x$ *does not converge to zero as* $k \to +\infty$*, but* $\lim_{n\to+\infty} T_{k_n}x = 0$*.*

In the case that $\tilde{X} = X$*, then we also say that* x *is a Li-Yorke (semi-)irregular vector for* $(T_k)_{k\in\mathbb{N}}$*. The notions of a* $\tilde{X}$*-Li-Yorke (semi-)irregular vector and a Li-Yorke (semi-)irregular vector for the operator* $T : D(T) \to Y$ *are introduced similarly.*

It can be simply verified with the help of translation-invariance of metric $d_Y(\cdot,\cdot)$ and the properties (2)-(4) that if $0 \neq x \in \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ is a $\tilde{X}$-Li-Yorke irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T), then $(T_k)_{k\in\mathbb{N}}$ (T) is $\tilde{X}$-Li-Yorke chaotic, with $S \equiv span\{x\}$ being the corresponding scrambled set. On the other hand, if the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T) is $\tilde{X}$-Li-Yorke chaotic and S is the corresponding $\tilde{X}$-scrambled set, then for each two distinct points $x,\ y \in S$, $x - y$ is a $\tilde{X}$-Li-Yorke semi-irregular vector for $(T_k)_{k\in\mathbb{N}}$ (the operator T).

Let $X' \subseteq \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ be a linear manifold. Then it is said that X' is a $\tilde{X}$-Li-Yorke irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T) (Li-Yorke irregular manifold for $(T_k)_{k\in\mathbb{N}}$ (T), in the case that $\tilde{X} = X$) iff any vector $x \in X' \setminus \{0\}$ is a $\tilde{X}$-Li-Yorke irregular vector for $(T_k)_{k\in\mathbb{N}}$ (T); the notion of a ($\tilde{X}$-)Li-Yorke semi-irregular manifold for $(T_k)_{k\in\mathbb{N}}$ (T) is defined similarly. Further on, it is said that X' is a uniformly $\tilde{X}$-Li-Yorke irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T) (uniformly Li-Yorke irregular manifold for $(T_k)_{k\in\mathbb{N}}$ (T), in the case that $\tilde{X} = X$) iff there exists $q \in \mathbb{N}$ such that, for every $x \in X' \setminus \{0\}$, there exists a sequence (k_n) of non-negative integers such that the set $\{p_q^Y(T_k x) : k \in \mathbb{N}\}$ is unbounded in $[0,\infty)$ and $\lim_{n\to+\infty} T_{k_n}x = 0$; this notion can be similarly introduced for ($\tilde{X}$-)Li-Yorke semi-irregular vectors. It is evident that if $0 \neq x \in \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ is a $\tilde{X}$-Li-Yorke irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$ (the operator T), then $X' \equiv span\{x\}$ is a uniformly $\tilde{X}$-Li-Yorke (semi-)irregular manifold for $(T_k)_{k\in\mathbb{N}}$ (T).

If $(T_k)_{k\in\mathbb{N}}$ and $\tilde{X}$ are given in advance, then we define the linear mappings $\mathcal{T}_k : D(\mathcal{T}_k) \to Y$ by $D(\mathcal{T}_k) := D(T_k) \cap \tilde{X}$ and $\mathcal{T}_k x := T_k x$, $x \in D(\mathcal{T}_k)$ ($k \in \mathbb{N}$). Then $(\mathcal{T}_k)_{k\in\mathbb{N}}$ is a sequence of linear mappings between the Fréchet spaces $\tilde{X}$ and Y. The following proposition shows that $\tilde{X}$-Li-Yorke chaotic properties of sequence $(T_k)_{k\in\mathbb{N}}$ can be analysed by passing to sequence $(\mathcal{T}_k)_{k\in\mathbb{N}}$; the proof is very easy and therefore omitted. Similar statement can be formulated for $\tilde{X}$-Li-Yorke chaotic properties of unbounded linear operators.

Proposition 2.3. *(i) The sequence* $(T_k)_{k\in\mathbb{N}}$ *is (densely)* $\tilde{X}$*-Li-Yorke chaotic iff the sequence* $(\mathcal{T}_k)_{k\in\mathbb{N}}$ *is (densely) Li-Yorke chaotic.*

(ii) A vector x *is a* $\tilde{X}$*-Li-Yorke (semi-)irregular vector for the sequence* $(T_k)_{k\in\mathbb{N}}$ *iff* x *is a Li-Yorke (semi-)irregular vector for the sequence* $(\mathcal{T}_k)_{k\in\mathbb{N}}$*.*

(iii) A linear manifold $X' \subseteq \bigcap_{k=1}^{\infty} D(T_k) \cap \tilde{X}$ *is a (uniformly)* $\tilde{X}$*-Li-Yorke (semi-)irregular manifold for the sequence* $(T_k)_{k\in\mathbb{N}}$ *iff* X' *is a (uniformly) Li-Yorke (semi-)irregular manifold for the sequence* $(\mathcal{T}_k)_{k\in\mathbb{N}}$*.*

In order to relax our further exposition, we will assume that $\tilde{X} = X$ henceforth. The following theorem provides generalizations of [7, Proposition 3 and 5; Corollary 4 and 6]; the proof can be deduced by using the arguments contained in the proofs of these assertions and therefore omitted.

Theorem 2.4. *Supppose that $T_k \in L(X, Y)$ for all $k \in \mathbb{N}$.*

(i) The set of all vectors $x \in X$ such that $(T_k x)_{k\in\mathbb{N}}$ has a subsequence converging to zero is a G_δ-set in X.

(ii) If the set of all points $x \in X$ such that $(T_k x)_{k\in\mathbb{N}}$ has a subsequence converging to zero is dense in X, then it is residual in X.

(iii) If the set $\{T_k u : k \in \mathbb{N}\}$ is unbounded in Y, then the sequence $(T_k)_{k\in\mathbb{N}}$ has a residual set of vectors with unbounded orbits, i.e., the vectors $x \in X$ for which the set $\{T_k x : k \in \mathbb{N}\}$ is unbounded in Y.

(iv) If the set of all irregular vectors for the sequence $(T_k)_{k\in\mathbb{N}}$ is dense in X, then it is residual in X.

Unfortunately, the assertions of [7, Lemma 7, Theorem 8, Lemma 13] cannot be so easily reconsidered for sequences of operators, even in the case that they are all continuous and $X = Y$. The semigroup property of sequence $(T^k)_{k\in\mathbb{N}}$, where $T \in L(X)$, plays an important role in the proof of [7, Lemma 7]. Concerning [7, Theorem 9], we can only prove the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$ of this theorem for sequences of linear, possibly unbounded, linear operators: summa summarum, we are not in a position to apply the method from our previous research of distributionally chaotic properties of linear operators [9] (cf. also [6]) in the analysis of Li-Yorke chaotic properties of fractionally integrated C-semigroups and abstract fractional differential equations.

In the above-mentioned paper, we have already constructed some important examples of unbounded differential operators that are Li-Yorke chaotic:

Example 2.5. *Suppose that X is separable, $D(A)$ and $R(C)$ are dense in X, $CA \subseteq AC$, $z_0 \in \mathbb{C} \setminus \{0\}$, $\beta \geq -1$, $d \in (0, 1]$, $m \in (0, 1)$, $\varepsilon \in (0, 1]$ and $\gamma > -1$. Denote $B_d = \{z \in \mathbb{C} : |z| \leq d\}$. Assume, further, that the following conditions hold:*

(i) $P_{z_0,\beta,\varepsilon,m} := e^{i\arg(z_0)}\Big(|z_0| + (P_{\beta,\varepsilon,m} \cup B_d)\Big) \subseteq \rho_C(A)$, $(\varepsilon, m(1 + \varepsilon)^{-\beta}) \in \partial B_d$,

(ii) the family $\{(1 + |\lambda|)^{-\gamma}(\lambda - A)^{-1}C : \lambda \in P_{z_0,\beta,\varepsilon,m}\}$ is equicontinuous,

(iii) the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, $\lambda \in P_{z_0,\beta,\varepsilon,m}$ is continuous for every fixed element $x \in X$, and

(iv) there exist a dense subset X_0 of X and a number $\lambda \in \sigma_p(A)$ such that $\lim_{k\to\infty} A^k x = 0$, $x \in X_0$ and $|\lambda| > 1$.

Then we have proved in [9, Theorem 3.13] that there exists a dense uniformly distributionally irregular manifold W for the operator zA^n, where $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $|z| = 1$. This, in particular, implies that the operator zA^n is densely Li-Yorke chaotic, with W being the corresponding scrambled set.

A concrete example can be simply constructed. Suppose that $r > 0$, $\sigma > 0$, $\nu = \sigma/\sqrt{2}$, $\gamma = r/\mu - \mu$, $s > 1$, $s\nu > 1$ and $\tau \geq 0$. Set

$$Y^{s,\tau} := \left\{u \in C((0, \infty)) : \lim_{x\to 0} \frac{u(x)}{1 + x^{-\tau}} = \lim_{x\to\infty} \frac{u(x)}{1 + x^s} = 0\right\}.$$

Then $Y^{s,\tau}$, equipped with the norm

$$\|u\|_{s,\tau} := \sup_{x>0} \left| \frac{u(x)}{\Big(1 + x^{-\tau}\Big)\Big(1 + x^s\Big)} \right|, \quad u \in Y^{s,\tau},$$

becomes a separable Banach space. Let $D_\mu := \nu x d/dx$, with maximal domain in $Y^{s,\tau}$, and let the Black-Scholes operator $\mathcal{B}$ be defined by $\mathcal{B} := D_\nu^2 + \gamma D_\mu - r$. Let us recall that H. Emamirad, G. R. Goldstein and J. A. Goldstein have proved in [11] that the operator $\mathcal{B}$ generates a chaotic strongly continuous semigroup (it can be easily seen that the Black-Scholes semigroup is densely distributionally chaotic, as well; cf. [9] for the notion). By [11, Lemma 3.3], the proof of [11, Lemma 3.5] (cf. especially the Figure 1 in the abovementioned paper, in the $Ox'y'$ coordinate system, with $x' = x/\nu$ and $y' = y/\nu$), it readily follows that the operator $A = \mathcal{B}$ satisfies the properties (i)-(iv) with $C = I$. Hence, $\mathcal{B}$ is densely Li-Yorke chaotic.

We close this section by observing that the arguments contained in the proof of [7, Proposition 11] can serve one to verify the validity of following proposition.

Proposition 2.6. *Suppose that the set consisted of all semi-irregular vectors of a linear densely defined operator T is dense in X. Then the operator T^* cannot have an eigenvalue λ with $|\lambda| \geq 1$.*

3. Li-Yorke chaotic properties of strongly continuous semigroups

An operator family $(T(t))_{t\geq 0}$ $(T(t) \in L(X),\ t \geq 0)$ is said to be a strongly continuous semigroup iff:

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s),\ t,\ s \geq 0$ and

(iii) the mapping $t \mapsto T(t)x,\ t \geq 0$ is continuous for every fixed $x \in X$.

The linear operator

$$A := \left\{(x, y) \in X \times X : \lim_{t\to 0+} \frac{T(t)x - x}{t} = y\right\}$$

is said to be the infinitesimal generator of $(T(t))_{t\geq 0}$.

The notions of various types of Li-Yorke chaos for strongly continuous semigroups on Banach spaces, introduced by Wu in [19, Definition 1.1], can be very simply transferred to Fréchet spaces. Here we will only consider the notions of (subspace) Li-Yorke chaoticity and (subspace) dense Li-Yorke chaoticity of strongly continuous semigroups; let us recall that $\tilde{X}$ denotes a closed linear subspace of X.

Definition 3.1. *A strongly continuous semigroup $(T(t))_{t\geq 0} \subseteq L(X)$ is said to be $\tilde{X}$-Li-Yorke chaotic iff there exists an uncountable set $S \subseteq \tilde{X}$ ($\tilde{X}$-scrambled set) such that for each pair $x,\ y \in S$ of distinct points we have that:*

$$\liminf_{t\to +\infty} d(T(t)x, T(t)y) = 0 \text{ and } \limsup_{t\to +\infty} d(T(t)x, T(t)y) > 0. \tag{6}$$

If we can choose S to be dense in $\tilde{X}$, then we say that $(T(t))_{t\geq 0}$ is densely $\tilde{X}$-Li-Yorke chaotic. Any pair (x, y) satisfying (6) is called a $\tilde{X}$-Li-Yorke pair for $(T(t))_{t\geq 0}$. Finally, if $\tilde{X} = X$, then we say that $(T(t))_{t\geq 0}$ is (densely) Li-Yorke chaotic and that (x, y) is a Li-Yorke pair for $(T(t))_{t\geq 0}$; in this case, S is called a scrambled set.

The notion of a $\tilde{X}$-Li-Yorke (semi-)irregular vector for $(T(t))_{t\geq 0}$ is introduced in the following definition.

Definition 3.2. *Let $(T(t))_{t\geq 0} \subseteq L(X)$ be a strongly continuous semigroup, and let $x \in \tilde{X}$. Then it is said that x is:*

(i) a $\tilde{X}$-Li-Yorke irregular vector for $(T(t))_{t\geq 0}$ iff there exists a sequence (t_n) in $[0, \infty)$ such that the set $\{T(t)x : t \geq 0\}$ is unbounded in X and $\lim_{n\to+\infty} T(t_n)x = 0$;

(ii) a $\tilde{X}$-Li-Yorke semi-irregular vector for $(T(t))_{t\geq 0}$ iff there exists a sequence (t_n) in $[0, \infty)$ such that $T(t)x$ does not converge to zero as $t \to +\infty$, but $\lim_{n\to+\infty} T(t_n)x = 0$.

In the case that $\tilde{X} = X$, then we also say that x is a Li-Yorke (semi-)irregular vector for $(T(t))_{t\geq 0}$.

Before proceeding further, we want to observe that, in the formulation of part (i), we do not require that $\{T(t)x : t \geq 0\}$ is a subset of $\tilde{X}$. The hypercyclicity of strongly continuous semigroups in Fréchet spaces is a stronger notion than the dense Li-Yorke chaoticity. Speaking-matter-of-factly, the continuous analogue of Herrero-Bourdon theorem [12, Theorem 7.17] holds for strongly continuous semigroups in Fréchet spaces; using this fact, it readily follows that any hypercyclic strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Fréchet space X has a dense subspace S consisting entirely of hypercyclic vectors, so that $(T(t))_{t\geq 0}$ is automatically densely Li-Yorke chaotic, with S being the corresponding scrambled set ([2], [12]). Regrettably, only a few statements from [16, Section 3.1] can be reformulated for Li-Yorke chaotic strongly continuous semigroups.

If $x \neq 0$ is a $\tilde{X}$-Li-Yorke irregular vector for $(T(t))_{t\geq 0}$, then $(T(t))_{t\geq 0}$ is $\tilde{X}$-Li-Yorke chaotic; on the other hand, if $(T(t))_{t\geq 0}$ is $\tilde{X}$-Li-Yorke chaotic and S is the corresponding $\tilde{X}$-scrambled set, then, for every two distinct vectors $x,\ y \in S$, $x-y$ is a $\tilde{X}$-Li-Yorke semi-irregular vector for $(T(t))_{t\geq 0}$. Suppose now that $X' \subseteq \tilde{X}$ is a linear manifold. In analogy with our previous considerations of Li-Yorke chaotic properties of unbounded linear operators and their sequences, we define the following notions: X' is called a $\tilde{X}$-Li-Yorke irregular manifold for $(T(t))_{t\geq 0}$ (Li-Yorke irregular manifold for $(T(t))_{t\geq 0}$, in the case that $\tilde{X} = X$) iff any vector $x \in X' \setminus \{0\}$ is a $\tilde{X}$-Li-Yorke irregular vector for $(T(t))_{t\geq 0}$; the notion of $\tilde{X}$-Li-Yorke semi-irregular manifold for $(T(t))_{t\geq 0}$ (Li-Yorke semi-irregular manifold for $(T(t))_{t\geq 0}$) is defined similarly. Clearly, if X' is a $\tilde{X}$-Li-Yorke irregular manifold for $(T(t))_{t\geq 0}$, then X' is a $\tilde{X}$-scrambled set. It is said that X' is a uniformly $\tilde{X}$-Li-Yorke irregular manifold for $(T(t))_{t\geq 0}$ (uniformly Li-Yorke irregular manifold for $(T(t))_{t\geq 0}$, in the case that $\tilde{X} = X$) iff there exists $q \in \mathbb{N}$ such that, for every $x \in X' \setminus \{0\}$, there exists a sequence (t_n) in $[0, \infty)$ such that the set $\{p_q(T(t)x) : t \geq 0\}$ is unbounded in $[0, \infty)$ and $\lim_{n\to+\infty} T(t_n)x = 0$; this notion can be similarly introduced for $\tilde{X}$-Li-Yorke semi-irregular vectors. Clearly, if $0 \neq x \in \tilde{X}$ is a $\tilde{X}$-Li-Yorke (semi-)irregular vector for $(T(t))_{t\geq 0}$, then $X' \equiv span\{x\}$ is a uniformly $\tilde{X}$-Li-Yorke (semi-)irregular manifold for $(T(t))_{t\geq 0}$.

The conjugacy lemma for Li-Yorke chaos of strongly continuous semigroups reads as follows. A straightforward proof is omitted.

Theorem 3.3. *Supppose that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X, $\tilde{X}$ is a closed linear subspace of X, $(S(t))_{t\geq 0}$ is a strongly continuous semigroup in Y, $\Phi : X \to Y$ is a linear continuous isomorphism and $S(t)\circ\Phi = \Phi\circ T(t)$ for all $t \geq 0$. Then the following holds:*

(i) $(T(t))_{t\geq 0}$ is (densely) $\tilde{X}$-Li-Yorke chaotic iff $(S(t))_{t\geq 0}$ is (densely) $\Phi(\tilde{X})$-Li-Yorke chaotic.

(ii) An element $x \in \tilde{X}$ is a $\tilde{X}$-Li-Yorke (semi-)irregular vector for $(T(t))_{t\geq 0}$ iff $\Phi(x)$ is a $\Phi(\tilde{X})$-Li-Yorke (semi-)irregular vector for $(S(t))_{t\geq 0}$.

It can be simply verified that the following continuous version of Theorem 2.4 holds good.

Theorem 3.4. *Supppose that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X.*

(i) The set of all vectors $x \in \tilde{X}$ such that there exists a sequence (t_n) in $[0, \infty)$ with $\lim_{n\to+\infty} T(t_n)x = 0$ is a G_δ-set in the Fréchet space $\tilde{X}$.

(ii) If the set of all points $x \in \tilde{X}$ such that there exists a sequence (t_n) in $[0, \infty)$ with $\lim_{n\to+\infty} T(t_n)x = 0$ is dense in $\tilde{X}$, then it is residual in $\tilde{X}$.

(iii) If there exists $u \in \tilde{X}$ such that the set $\{T(t)u : t \geq 0\}$ is unbounded in X, then $(T(t))_{t\geq 0}$ has a residual set of vectors with unbounded orbits in the Fréchet space $\tilde{X}$, i.e., the vectors $x \in \tilde{X}$ for which the set $\{T(t)x : t \geq 0\}$ is unbounded in X.

(iv) If the set of all irregular vectors for $(T(t))_{t\geq 0}$ is dense in $\tilde{X}$, then it is residual in $\tilde{X}$.

Now we will transfer the assertion of [7, Lemma 7] to strongly continuous semigroups in Fréchet spaces.

Lemma 3.5. *Suppose that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X. If $x \in X$ is a semi-irregular vector for $(T(t))_{t\geq 0}$ which is not irregular for $(T(t))_{t\geq 0}$, then there exists a sequence (x_j) of non-zero vectors in X such that $\alpha x + \sum_{j=1}^{\infty} \beta_j x_j$ is an irregular vector for $(T(t))_{t\geq 0}$, whenever α is a scalar and (β_j) is a sequence of scalars that takes only finitely many values and has infinitely many non-zero coordinates.*

Proof. The proof is very similar to that of [7, Lemma 7] and, because of that, we will only outline a few most important details. Since the function $t \mapsto T(t)x$, $t \geq 0$ does not converge to 0 as $t \to +\infty$, there exist a sequence (t_n) in $[0, \infty)$ and an absolutely convex closed neighborhood V of 0 in X such that $T(t_n)x \notin V$ for infinitely many values of n. On the other hand, the set $\{T(t)x : t \geq 0\}$ must be bounded in X, whence we may conclude that there exists $r \in \mathbb{N}$ such that $T(t)x \in rV$ for all $t \geq 0$. Since x is a semi-irregular vector for $(T(t))_{t\geq 0}$, we have the existence of a sequence (t_n') in $[0, \infty)$ such that $\lim_{n\to\infty} T(t_n')x = 0$. Let (a_n) be a strictly increasing sequence in $[0,\infty)$ satisfying that $\lim_{n\to\infty} a_n = +\infty$ and $a_n > t_n + t'_n$ for all $n \in \mathbb{N}$. Using the local equicontinuity of $(T(t))_{t\geq 0}$ and the proof of afore-mentioned lemma, we may construct a sequence $(V_j)_{j\in\mathbb{N}_0}$ of absolutely convex and closed neighborhoods of 0 in X, an increasing sequence $(c_k)_{k\in\mathbb{N}_0}$ of non-negative integers satisfying $c_0 = 0$, $c_k = k^2(2 + r(c_0 + \cdots + c_{k-1}))$ for $k \geq 1$, and three sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}_0}$, $(p_k)_{k\in\mathbb{N}_0}$ of positive integers, so that the following conditions hold:

(i) $V_0 = V$, $V_j + V_j \subseteq V_{j-1}$ and $V_j \subseteq V_{j-1}$ for all $j \in \mathbb{N}$,

(ii) $V_p + V_{p+1} + \cdots + V_q \subseteq V_{p-1}$ whenever $1 \leq p \leq q$,

(iii) $T(t)(V_j) \subseteq V_{j-n}$ whenever $0 \leq j \leq n$ and $0 \leq t \leq a_{j-n+1} + \cdots + a_j$,

(iv) $n_1 < m_1 < n_2 < m_2 < \cdots$, $p_1 < p_2 < \cdots$, $m_0 = p_0 = 0$,

(v) $T(t'_{n_k})x \in c_k^{-1} V_{m_{k-1}+p_{k-1}}$,

(vi) $T(t_{m_k})x \in V$,

(vii) $T(t'_{p_k})x \in V_k$, and

(viii) $T(t'_{p_k}) \sum_{j=1}^{k} \lambda_j c_j T(t'_{n_k})x \in V_k$ whenever $k \in \mathbb{N}$ and $|\lambda_j| \leq k$, $j = 1, 2, \cdots, k$.

Set $x_j := c_j T(t_{n_j})x$ ($j \in \mathbb{N}$) an fix an arbitrary sequence (β_j) of scalars that takes only finitely many values and has infinitely many non-zero coordinates. Using the argumentation contained in the proof of [7, Lemma 7], we can prove that the limit $y := \sum_{j=1}^{\infty} \beta_j x_j$ exists in X as well as that $T(t'_{p_k})y \to 0$ as $k \to +\infty$ and $T(t_{m_k - n_k})y \notin kV$ for all sufficiently large values of k with $\beta_k \neq 0$. Hence, y is a Li-Yorke irregular vector for $(T(t))_{t\geq 0}$. The remaining part of proof is simple and therefore omitted. □

Remark 3.6. *Suppose that $\tilde{X}$ is a closed linear subspace of X and $x \in \tilde{X}$ is a semi-irregular vector for $(T(t))_{t\geq 0}$ which is not irregular for $(T(t))_{t\geq 0}$. If, additionally, $T(t)x \in \tilde{X}$ for all $t \geq 0$, then $(x_j = c_j T(t_{n_j})x)_{j\in\mathbb{N}}$ is a sequence in $\tilde{X}$.*

Keeping Lemma 3.5 and Remark 3.6 in mind, it is straightforward to prove the following continuous analogons of [7, Theorem 8, Theorem 9]:

Theorem 3.7. *Suppose that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X. If x is a $\tilde{X}$-Li-Yorke semi-irregular vector for $(T(t))_{t\geq 0}$ and $T(t)x \in \tilde{X}$ for all $t \geq 0$, then any neighborhood of x in $\tilde{X}$ contains a $\tilde{X}$-Li-Yorke irregular vector for $(T(t))_{t\geq 0}$.*

Theorem 3.8. *Suppose that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X. Consider the following assertions:*

(i) *$(T(t))_{t\geq 0}$ is $\tilde{X}$-Li-Yorke chaotic.*

(ii) *$(T(t))_{t\geq 0}$ admits a $\tilde{X}$-Li-Yorke pair.*

(iii) *$(T(t))_{t\geq 0}$ admits a $\tilde{X}$-Li-Yorke semi-irregular vector.*

(iv) *$(T(t))_{t\geq 0}$ admits a $\tilde{X}$-Li-Yorke irregular vector.*

Then we have (i) ⇒ (ii) ⇒ (iii) and (iv) ⇒ (i). Furthermore, if $T(t)(\tilde{X}) \subseteq \tilde{X}$ for all $t \geq 0$, then the above is equivalent.

In connection with Theorem 3.8, it should be said that it is already known that the existence of one scrambled pair implies Li-Yorke on the interval and on graphs.

In the subsequent theorem, we will reconsider the assertion of [7, Theorem 10] for strongly continuous semigroups in Fréchet spaces. The proof can be deduced as in discrete case and therefore omitted.

Theorem 3.9. *Suppose that X is separable and $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X. Consider the following assertions:*

(i) $(T(t))_{t\geq 0}$ *admits a dense set of* $\tilde{X}$*–Li-Yorke irregular vectors.*

(ii) $(T(t))_{t\geq 0}$ *admits a residual set of* $\tilde{X}$*–Li-Yorke irregular vectors.*

(iii) $(T(t))_{t\geq 0}$ *is densely* $\tilde{X}$*-Li-Yorke chaotic.*

(iv) $(T(t))_{t\geq 0}$ *admits a dense set of* $\tilde{X}$*–Li-Yorke semi-irregular vectors.*

Then we have (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv). Furthermore, if* $T(t)(\tilde{X}) \subseteq \tilde{X}$ *for all* $t \geq 0$*, then the above is equivalent.*

In our previous considerations, we have seen that the inclusion $T(t)(\tilde{X}) \subseteq \tilde{X}$ $(t \geq 0)$ is crucial for the validity of a great number of assertions concerning the subspace Li-Yorke chaoticity of strongly continuous semigroups in Fréchet spaces. The assertions of [7, Lemma 13, Theorem 15, Theorem 17, Theorem 20, Corollary 21] can be transferred to operator semigroups in Fréchet spaces without any substantial difficulties, as well, but then it is almost inevitable to assume that the above inclusion holds (we will omit the proofs of Proposition 3.11, Theorem 3.12-Theorem 3.13 and Corollary 3.14 below which correspond to these assertions). If so, $(T(t)_{|\tilde{X}})_{t\geq 0}$ is a strongly continuous semigroup in the Fréchet space $\tilde{X}$ and we do not obtain anything new and relevant compared with the case in which $\tilde{X} = X$. Together with the separability of state space X, the assumption $\tilde{X} = X$ will be standing henceforth.

Definition 3.10. *Suppose that* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup in* X*.*

(i) We say that $(T(t))_{t\geq 0}$ *satisfies the Li-Yorke chaos criterion iff there exists a subset* X_0 *of* X *with the following properties:*

   (a) to every $x \in X_0$ *there exists a sequence* (t_n) *in* $[0, \infty)$ *such that* $\lim_{n\to\infty} T(t_n)x = 0$*,*

   (b) there is a bounded sequence (a_n) *in* $\overline{span(X_0)}$ *such that the set* $\{T(t)a_n : t \geq 0\}$ *is unbounded.*

(ii) We say that $(T(t))_{t\geq 0}$ *satisfies the dense Li-Yorke chaos criterion iff there exists a dense subset* X_0 *of* X *with the properties (a) and (b) clarified above.*

Proposition 3.11. *Suppose that* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup in* X *and there exists a subset* X_0 *of* X *with the following properties:*

(a) $\lim_{t\to\infty} T(t)x = 0$, $x \in X_0$,

(b) there is a bounded sequence (a_n) *in* $\overline{span(X_0)}$ *such that the set* $\{T(t)a_n : t \geq 0\}$ *is unbounded.*

Then there exists a sequence (x_j) *of non-zero vectors in* X *such that* $\sum_{j=1}^{\infty} \beta_j x_j$ *is an irregular vector for* $(T(t))_{t\geq 0}$*, whenever* (β_j) *is a sequence of scalars that takes only finitely many values and has infinitely many non-zero coordinates.*

Theorem 3.12. *Suppose that* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup in* X*. Then* $(T(t))_{t\geq 0}$ *is (densely) Li-Yorke chaotic iff it satisfies the (dense) Li-Yorke chaos criterion.*

Theorem 3.13. *Suppose that* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup in* X *and there exists a dense subset* X_0 *of* X *with the following properties:*

(a) $\lim_{t\to\infty} T(t)x = 0$, $x \in X_0$,

(b) there is a bounded sequence (a_n) *in* X *such that the set* $\{T(t)a_n : t \geq 0\}$ *is unbounded.*

Then $(T(t))_{t\geq 0}$ *admits a dense Li-Yorke chaotic irregular manifold.*

Corollary 3.14. *Suppose that* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup in* X *and there exists a dense subset* X_0 *of* X *with the property that* $\lim_{t\to\infty} T(t)x = 0$, $x \in X_0$*. Then the following assertions are equivalent:*

(a) $(T(t))_{t\geq 0}$ *is Li-Yorke chaotic.*

(b) $(T(t))_{t\geq 0}$ *admits a dense Li-Yorke irregular manifold.*

(c) $(T(t))_{t\geq 0}$ *admits an unbounded orbit.*

From our previous analyses, it readily follows that the Li-Yorke chaoticity of operator $T(t_0)$ for some number $t_0 > 0$ implies the Li-Yorke chaoticity of strongly continuous semigroup $(T(t))_{t\geq 0}$ in X.

We would like to propose the following problem:

Problem 1. Suppose that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X. Is it true that $(T(t))_{t\geq 0}$ is Li-Yorke chaotic iff for each/some number $t_0 > 0$ the operator $T(t_0)$ is Li-Yorke chaotic?

The affirmative answer would imply that there is no immediately compact semigroup $(T(t))_{t\geq 0}$ in a Fréchet space X that is Li-Yorke chaotic (we define this notion as in the Banach space case; cf. also [7, Proposition 18], [5, Corollary 6] and [15, Theorem 4.12-Corollary 4.15]). The study of disjoint Li-Yorke chaoticity of strongly continuous semigroups will be taken up somewhere else.

4. Li-Yorke chaotic properties of translation semigroups and strongly continuous semigroups induced by semiflows

In this section, we investigate the Li-Yorke chaotic properties of translation semigroups and strongly continuous semigroups induced by semiflows. Suppose that Δ is $[0,\infty)$ or $\mathbb{R}$. A measurable function $\rho : \Delta \to (0,\infty)$ is said to be an *admissible weight function* iff there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(t) \leq Me^{\omega|t'|}\rho(t+t')$ for all $t,\ t' \in \Delta$. For such a function ρ, we introduce the following Banach spaces:

$$L^p_\rho(\Delta) := \Big\{u : \Delta \to \mathbb{C} \mid u \text{ is measurable and } \|u\|_p < \infty\Big\},$$

where $p \in [1,\infty)$ and $\|u\|_p := (\int_\Delta |u(t)|^p \rho(t)\,dt)^{1/p}$, as well as

$$C_{0,\rho}(\Delta) := \Big\{u : \Delta \to \mathbb{C} \mid u \text{ is continuous and } \lim_{t\to\infty} u(t)\rho(t) = 0\Big\},$$

with $\|u\| := \sup_{t\in\Delta} |u(t)\rho(t)|$. Set $(T(t)f)(x) := f(x+t)$, $x \in \Delta$, $t \in \Delta$. Then it is well known that $(T(t))_{t\geq 0}$ is a C_0-semigroup on $L^p_\rho([0,\infty))$ and $C_{0,\rho}([0,\infty))$, and that $(T(t))_{t\in\mathbb{R}}$ is a C_0-group on $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$; see [10, Definition 4.3, Lemma 4.6, Theorem 4.9].

Up to now, we have only one result on the Li-Yorke chaoticity of translation semigroups in weighted function spaces, proved by Wu in [19], which states that the strongly continuous translation semigroup $(T(t))_{t\geq 0}$ is Li-Yorke chaotic on $L^p_\rho([0,\infty))$ iff $\liminf_{t\to+\infty} \rho(t) = 0$, provided in advance that the function ρ is bounded from above (cf. [19, Corollary 2.3]). In this case, the hypercyclicity of $(T(t))_{t\geq 0}$ is equivalent with its Li-Yorke chaoticity by [10, Theorem 4.7]. On the other hand, if the function ρ is not bounded from above, then it is well known that there exists a strongly continuous translation semigroup that is Li-Yorke chaotic but not hypercyclic (cf. [19, Example 2.4] and [3, Example 2.7]).

Unfortunately, the result of Wu [19] does not give us a necessary and sufficient condition for $(T(t))_{t\geq 0}$ to be Li-Yorke chaotic on $L^p_\rho([0,\infty))$. In the next theorem, we will completely profile the (dense) Li-Yorke chaoticity of $(T(t))_{t\geq 0}$ in terms of weight function $\rho(t)$.

Theorem 4.1. *Let* $\rho : [0,\infty) \to (0,\infty)$ *be an admissible weight function.*

(i) The strongly continuous translation semigroup $(T(t))_{t\geq 0}$ *is (densely) Li-Yorke chaotic on* $L^p_\rho([0,\infty))$ *iff*

$$\limsup_{t\to+\infty} \left\| \frac{\chi_{(0,t)}(\cdot)\rho(\cdot - t)}{\rho(\cdot)} \right\|_{L^\infty([0,\infty),\rho(x)dx)} = +\infty. \tag{7}$$

(ii) The strongly continuous translation semigroup $(T(t))_{t\geq 0}$ *is (densely) Li-Yorke chaotic on* $C_{0,\rho}([0,\infty))$ *iff*

$$\limsup_{t\to+\infty} \inf_{C>0} \left\{ \frac{\rho(x)}{\rho(x+t)} \leq C \text{ for all } x \geq 0 \right\} = +\infty. \tag{8}$$

Proof. It is well-known that $C_c((0,\infty))$, the space which consists of all continuous functions $f:(0,\infty)\to\mathbb{C}$ with compact support, is dense in both spaces $L^p_\rho([0,\infty))$ and $C_{0,\rho}([0,\infty))$. Clearly, for every $f\in X_0 := C_c((0,\infty))$, we have that $\lim_{t\to+\infty}T(t)f=0$. Hence, Corollary 3.14 implies that $(T(t))_{t\geq 0}$ is (densely) Li-Yorke chaotic iff $(T(t))_{t\geq 0}$ admits an unbounded orbit. Using the uniform boundedness principle, it readily follows that $(T(t))_{t\geq 0}$ is not (densely) Li-Yorke chaotic iff there exists a finite constant $M\geq 1$ such that $\|T(t)\|\leq M$, $t\geq 0$. Since the norm of $T(t)$ is equal to the expression appearing on the left hand side of (7), resp. (8), the result is proved (cf. [14, Theorem 3.2, Proposition 3.11] for the case in which $X=L^p_\rho([0,\infty))$, and [14, Theorem 2.2] for the case in which $X=C_{0,\rho}((0,\infty))$ and $\rho(t)$ is upper semicontinuous for $t>0$). □

Following the method proposed by Takeko [18], the author of this paper has considered in [16, Section 3.1] various hypercyclic and topologically mixing properties of strongly continuous semigroups on the function spaces $L^p([0,\infty):\mathbb{C})$ and $C_0([0,\infty):\mathbb{C})$. A strongly continuous semigroup $(T(t))_{t\geq 0}$ under consideration takes the form $(T(t)f)(x)=g(x,t)f(x+t)$, $x,\ t\geq 0$, $f\in X$, where $g:[0,\infty)\times[0,\infty)\to\mathbb{C}$ and satisifes the conditions (HT1)-(HT4) from [16, Lemma 3.1.22]. Taking into account Theorem 3.4, Theorem 4.1, as well as [16, Lemma 3.1.23] and the proof of [16, Theorem 3.1.25], we can simply clarify the necessary and sufficient conditions stating when the strongly continuous semigroup $(T(t))_{t\geq 0}$ of form described above is Li-Yorke chaotic on $L^p([0,\infty):\mathbb{C})$ and $C_0([0,\infty):\mathbb{C})$ (cf. also [16, Theorem 3.1.27, Example 3.1.28(iii)] and [14, Section 4]). Details are left to the interested reader.

If the state space X is $L^p_\rho(\mathbb{R})$ or $C_{0,\rho}(\mathbb{R})$, then a characterization of Li-Yorke chaoticity of translation semigroup $(T(t))_{t\geq 0}$ in terms of admissible weight function $\rho(t)$ is very difficult to be given. In connection with this problem, we would like to point out that Barrachina has constructed an example of admissible weight function $\rho:\mathbb{R}\to(0,\infty)$ such that $(T(t))_{t\geq 0}\subseteq L(L^p_\rho(\mathbb{R}))$ is both non-hypercyclic and completely distributionally chaotic (cf. [2, Definition 2.2, Example 5.4, Remark 5.2]), which implies in particular that any function from $L^p_\rho(\mathbb{R})$ is a Li-Yorke irregular vector for $(T(t))_{t\geq 0}$. Hence, $(T(t))_{t\geq 0}$ is both non-hypercyclic and completely Li-Yorke chaotic, i.e., the whole space $L^p_\rho(\mathbb{R})$ is a scrambled set for $(T(t))_{t\geq 0}$.

The hypercyclicity of strongly continuous semigroups induced by semiflows has been analysed for the first time by Kalmes in [14]-[15]. He dealt with the space $L^p(X,\mu)$, resp. $C_{0,\rho}(X)$, where X is a locally compact, σ-compact Hausdorff space, $p\in[1,\infty)$ and μ is a locally finite Borel measure on X, resp., X is a locally compact, Hausdorff space and $\rho:X\to(0,\infty)$ is an upper semicontinuous function. In this paper, we will consider the spaces $L^p_{\rho_1}(\Omega)$ and $C_{0,\rho}(\Omega)$, where Ω is an open non-empty subset of $\mathbb{R}^n$. Here, $\rho_1:\Omega\to(0,\infty)$ is a locally integrable function, the norm of an element $f\in L^p_{\rho_1}(\Omega)$ is given by $\|f\|_p := (\int_\Omega |f(x)|^p\rho_1(x)\,dx)^{1/p}$ and dx denotes Lebesgue's measure on $\mathbb{R}^n$. Recall that, for a given upper semicontinuous function $\rho:\Omega\to(0,\infty)$, the space $C_{0,\rho}(\Omega)$ consists of all continuous functions $f:\Omega\to\mathbb{C}$ satisfying that, for every $\epsilon>0$, $\{x\in\Omega : |f(x)|\rho(x)\geq\epsilon\}$ is a compact subset of Ω; equipped with the norm $\|f\| := \sup_{x\in\Omega}|f(x)|\rho(x)$, $C_{0,\rho}(\Omega)$ becomes a Banach space. Put, by common consent, $\sup_{x\in\emptyset}\rho(x):=0$ and denote by $C_c(\Omega)$ the space of all continuous functions $f:\Omega\to\mathbb{C}$ whose support is a compact subset of Ω. It is well known that $C_c(\Omega)$ is dense in both spaces, $L^p_{\rho_1}(\Omega)$ and $C_{0,\rho}(\Omega)$.

Suppose $n\in\mathbb{N}$, Ω is an open non-empty subset of $\mathbb{R}^n$ and Δ is $[0,\infty)$ or $\mathbb{R}$. A continuous mapping $\varphi:\Delta\times\Omega\to\Omega$ is called a *semiflow* [14]-[15] iff $\varphi(0,x)=x$, $x\in\Omega$,

$$\varphi(t+s,x)=\varphi(t,\varphi(s,x)),\ t,\ s\in\Delta,\ x\in\Omega \text{ and}$$

$$x\mapsto\varphi(t,x) \text{ is injective for all } t\in\Delta.$$

Denote by $\varphi(t,\cdot)^{-1}$ the inverse mapping of $\varphi(t,\cdot)$, i.e.,

$$y=\varphi(t,x)^{-1} \text{ iff } x=\varphi(t,y),\ t\in\Delta.$$

Given a number $t\in\Delta$, a semiflow $\varphi:\Delta\times\Omega\to\Omega$ and a function $f:\Omega\to\mathbb{C}$, define $T_\varphi(t)f:\Omega\to\mathbb{C}$ by $(T_\varphi(t)f)(x) := f(\varphi(t,x))$, $x\in\Omega$. Then $T_\varphi(0)f=f$, $T_\varphi(t)T_\varphi(s)f=T_\varphi(s)T_\varphi(t)f=T_\varphi(t+s)f$, $t,\ s\in\Delta$ and Brouwer's theorem implies $C_c(\Omega)\subseteq T_\varphi(t)[C_c(\Omega)]$. We refer the reader to [14, Theorem 2.1], resp. [14, Theorem 2.2], for the necessary and sufficient conditions stating when the composition operator $T_\varphi(t):L^p_{\rho_1}(\Omega)\to L^p_{\rho_1}(\Omega)$, resp. $T_\varphi(t):C_{0,\rho}(\Omega)\to C_{0,\rho}(\Omega)$, is well defined and continuous. The question whether $(T_\varphi(t))_{t\in\Delta}$ is a C_0-semigroup in $L^p_{\rho_1}(\Omega)$ or $C_{0,\rho}(\Omega)$ has been analyzed in [14] and [16]. We have the following:

Lemma 4.2. *Suppose $\varphi : \Delta\times\Omega \to \Omega$ is a semiflow and $\varphi(t,\cdot)$ is a continuously differentiable function for all $t \in \Delta$. Then $(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $L^p_{\rho_1}(\Omega)$ iff the following holds:*

$$\exists M,\ \omega \in \mathbb{R}\ \ \forall t \in \Delta : \rho_1(x) \leq Me^{\omega|t|}\rho_1(\varphi(t,x))|\det D\varphi(t,x)|\ a.e.\ x \in \Omega. \tag{9}$$

Lemma 4.3. *Let $\varphi : \Delta\times\Omega \to \Omega$ be a semiflow. Then $(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ iff the following holds:*

(i) $\exists M,\ \omega \in \mathbb{R}\ \ \forall t \in \Delta,\ x \in \Omega : \rho(x) \leq Me^{\omega|t|}\rho(\varphi(t,x))$ and

(ii) for every compact set $K \subset \Omega$ and for every $\delta > 0$ and $t \in \Delta$:

$$\varphi(t,\cdot)^{-1}(K)\ \cap\ \{x \in \Omega : \rho(x) \geq \delta\} \text{ is a compact subset of } \Omega. \tag{10}$$

We need to introduce the following condition:

(D) For every compact subset K of Ω, there exists $t_0 > 0$ such that $\varphi(t,\Omega) \cap K = \emptyset$, $t \geq t_0$.

Then, for every $f \in X_0 := C_c(\Omega)$, there exists $t_0 > 0$ such that $T_\varphi(t)f = 0$ for $t \geq t_0$. By the argumentation given in the proof of Theorem 4.1, it readily follows that the strongly continuous semigroup $(T_\varphi(t))_{t\geq 0}$ is not (densely) Li-Yorke chaotic iff there exists a finite constant $M \geq 1$ such that $\|T_\varphi(t)\| \leq M$, $t \geq 0$. Bearing in mind also [14, Theorem 2.1, Theorem 2.2] and Lemma 4.2-Lemma 4.3, the above implies:

Theorem 4.4. *(i) Suppose $\varphi : [0,\infty)\times\Omega \to \Omega$ is a semiflow, $\varphi(t,\cdot)$ is a continuously differentiable function for all $t \geq 0$, (D) holds, and (9) holds with $\Delta = [0,\infty)$. Set $C_t := \{x \in \Omega : Det\, D\varphi(t,\cdot) = 0\}$ and $\Omega_t := \Omega \setminus C_t$ $(t \geq 0)$. Then the strongly continuous semigroup $(T_\varphi(t))_{t\geq 0}$ is (densely) Li-Yorke chaotic on $L^p_{\rho_1}(\Omega)$ iff*

$$\limsup_{t\to+\infty}\left\| \frac{\chi_{\varphi(t,\Omega_t)}(\cdot)\rho_1(\varphi(-t,\cdot))Det\, D\varphi(-t,\cdot)}{\rho_1(\cdot)}\right\|_{L^\infty(\Omega,\rho_1(x)dx)} = +\infty.$$

(ii) Let $\varphi : [0,\infty)\times\Omega \to \Omega$ be a semiflow, let the conditions (i)-(ii) from Lemma 4.3 hold with $\Delta = [0,\infty)$, and let (D) hold. Then the strongly continuous semigroup $(T_\varphi(t))_{t\geq 0}$ is (densely) Li-Yorke chaotic on $C_{0,\rho}(\Omega)$ iff

$$\limsup_{t\to+\infty}\inf_{C>0}\left\{\frac{\rho(x)}{\rho(\varphi(t,x))} \leq C \text{ for all } x \in \Omega\right\} = +\infty.$$

There exists a great number of concrete examples in which the condition (D) does not hold; see e.g. [14, Example 3.19, Example 3.20]. We would like to address the problem of finding necessary and sufficient conditions characterizing the (dense) Li-Yorke chaoticity of $(T_\varphi(t))_{t\geq 0}$ on $L^p_{\rho_1}(\Omega)$ and $C_{0,\rho}(\Omega)$, in terms of weight ρ and semiflow φ.

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