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# Some inequalities for $\log \varphi$-convex functions

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**Abstract.** In this paper, we derive several Hermite-Hadamard type integral inequalities for log $\varphi$-convex functions. Our results represent refinement and improvement of the previously known results. Several special cases are discussed.

## 1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

named after C. Hermite and J. Hadamard. Inequality (1.1) can be considered as necessary and sufficient condition for a function to be convex. For useful details on Hermite-Hadamard type of integral inequalities (see [1,2,3,4,6,7,10,11,12,14,15,16]).

In recent years concept of convexity has been generalized and extended in several directions using novel and innovative ideas (see [2,4,5,8,10,12]). A significant generalization of classical convexity was the introduction of $\varphi$-convexity by Noor [10]. Noor [10] investigated various basic properties for the class of $\varphi$-convex function. Noor [8] extended Hermite-Hadamard type integral inequalities for $\varphi$-convex function. It is worth to mention here that $\varphi$-convex functions are nonconvex functions. In this paper, we consider the class of log $\varphi$-convex functions which was also introduced by Noor [10]. We derive several Hermite-Hadamard type inequalities for log $\varphi$-convex functions. Our results generalize several known results. The ideas and techniques used in the paper are interesting and may stimulate further research in this area.

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## 2. Preliminaries

In this section, we recall some basic results. Let $\mathbb{R}^n$ be a finite dimensional euclidian space, whose inner product and norm is denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $K_\varphi$ be a nonempty closed set in $\mathbb{R}^n$ and suppose $f, \varphi : K \rightarrow \mathbb{R}$ be continuous functions, where $0 \leq \varphi \leq \frac{\pi}{2}$.

**Definition 2.1 ([10]).** *Let $u \in K_\varphi$. Then the set $K_\varphi$ is said to be $\varphi$-convex, if*

$$u + te^{i\varphi}(v - u) \in K_\varphi, \quad \forall\, u, v \in K_\varphi, \ t \in [0,1].$$

For $\varphi = 0$, the set $K_\varphi$ reduces to the classical convex set $K$. That is,

$$u + t(v - u) \in K, \quad \forall u, v \in K, \ t \in [0,1].$$

**Remark 2.2 ([10]).** *Definition 2.1 says that there is a path starting from a point $u$ which is contained in $K_\varphi$. We do not require that the point $v$ should be one of the end point of the path. Note that, if we demand that $v$ should be an end point of the path for every pair of points $u, v \in K_\varphi$, then $e^{i\varphi}(v - u) = v - u$, if and only if $\varphi = 0$. Then the $\varphi$-convex $K_\varphi$ becomes the convex set $K$. It is clear that every convex set is a $\varphi$-convex set, but the converse is not necessarily true.*

**Definition 2.3 ([10]).** *A function $f$ on the $\varphi$-convex set $K_\varphi$ is said to be $\varphi$-convex with respect to $\varphi$, if*

$$f(u + te^{i\varphi}(v - u)) \leq (1 - t)f(u) + tf(v)\,, \quad \forall\, u, v \in K_\varphi, \ t \in [0,1].$$

For $\varphi = 0$, $\varphi$-convex function reduces to convex functions. This implies that every convex function is $\varphi$-convex function. However the converse is not true, see [14].

**Definition 2.4 ([10]).** *A function $f$ on the $\varphi$-convex set $K_\varphi$ is said to be* log *$\varphi$-convex with respect to $\varphi$, if*

$$f(u + te^{i\varphi}(v - u)) \leq [f(u)]^{1-t}[f(v)]^t\,, \quad \forall\, u, v \in K_\varphi, \ t \in [0,1].$$

***Remark 2.5.*** *From Definition 2.4, we have*

$$\log f(u + te^{i\varphi}(v - u)) \leq (1 - t)\log(f(u)) + t\log(f(v))\,, \quad \forall\, u, v \in K_\varphi, \ t \in [0,1].$$

*Using Remark 2.5, we have if $f$ is differentiable* log *$\varphi$-convex function.*

$$f(v) \geq f(u)\exp\left\langle \frac{f'_\varphi(u)}{f(u)}, v - u \right\rangle, \quad \forall u, v \in K_\varphi,$$

*where $f'_\varphi(.)$ is the $\varphi$-derivative of $f$, see [10].*

## 3. Main Results

In this section, we prove our main results.

**Theorem 3.1.** *Let $f, g : I = [a, a + e^{i\varphi}(b - a)] \rightarrow (0, \infty)$ be* log *$\varphi$-convex functions. If $\alpha + \beta = 1$, then*

$$\frac{1}{e^{i\varphi}(b - a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx$$
$$\leq \alpha\frac{f(a) + f(b)}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a))\right]^{\frac{1-\alpha}{\alpha}} + \beta\frac{g(a) + g(b)}{2}\left[L_{(\frac{1}{\beta}-1)}(g(b), g(a))\right]^{\frac{1-\beta}{\beta}}.$$

*Proof.* Let $f$ and $g$ be log $\varphi$-convex functions. Using inequality

$$xy \leq \alpha x^{\frac{1}{\alpha}} + \beta y^{\frac{1}{\beta}}, \quad \alpha, \beta > 0, \alpha + \beta = 1,$$

we have

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx = \int_0^1 f(a+te^{i\varphi}(b-a))g(a+te^{i\varphi}(b-a))$$

$$\leq \int_0^1 \left\{\alpha(f(a+te^{i\varphi}(b-a)))^{\frac{1}{\alpha}} + \beta(g(a+te^{i\varphi}(b-a)))^{\frac{1}{\beta}}\right\} dt$$

$$\leq \int_0^1 \left\{\alpha[(f(a))^{1-t}(f(b))^t]^{\frac{1}{\alpha}} + \beta[(g(a))^{1-t}(g(b))^t]^{\frac{1}{\beta}}\right\} dt$$

$$= \alpha(f(a))^{\frac{1}{\alpha}} \int_0^1 \left(\frac{f(b)}{f(a)}\right)^{\frac{t}{\alpha}} dt + \beta(g(a))^{\frac{1}{\beta}} \int_0^1 \left(\frac{g(b)}{g(a)}\right)^{\frac{t}{\beta}} dt$$

$$= \alpha^2(f(a))^{\frac{1}{\alpha}} \int_0^{\frac{1}{\alpha}} \left(\frac{f(b)}{f(a)}\right)^{u} du + \beta^2(g(a))^{\frac{1}{\beta}} \int_0^{\frac{1}{\beta}} \left(\frac{g(b)}{g(a)}\right)^{v} dv$$

$$= \alpha^2 \frac{(f(b))^{\frac{1}{\alpha}} - (f(a))^{\frac{1}{\alpha}}}{\log f(b) - \log f(a)} + \beta^2 \frac{(g(b))^{\frac{1}{\beta}} - (g(a))^{\frac{1}{\beta}}}{\log g(b) - \log g(a)}$$

$$= \alpha^2 \frac{(f(b))^{\frac{1}{\alpha}} - (f(a))^{\frac{1}{\alpha}}}{f(b) - f(a)} L(f(b), f(a)) + \beta^2 \frac{(g(b))^{\frac{1}{\alpha}} - (g(a))^{\frac{1}{\beta}}}{g(b) - g(a)} L(g(b), g(a))$$

$$= \alpha \left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a))\right]^{\frac{\alpha}{1-\alpha}} L(f(b), f(a)) + \beta \left[L_{(\frac{1}{\beta}-1)}(g(b), g(a))\right]^{\frac{\beta}{1-\beta}} L(g(b), g(a))$$

$$\leq \alpha \frac{f(a)+f(b)}{2} \left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a))\right]^{\frac{\alpha}{1-\alpha}} + \beta \frac{g(a)+g(b)}{2} \left[L_{(\frac{1}{\beta}-1)}(g(b), g(a))\right]^{\frac{\beta}{1-\beta}}.$$

This completes the proof. □

**Theorem 3.2.** *Let $f, g : I = [a, a + e^{i\varphi}(b-a)] \to (0, \infty)$ be* log *$\varphi$-convex functions with $a < a + e^{i\varphi}(b-a)$. Then, we have*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(2a + e^{i\varphi}(b-a) - x)dx$$

$$\leq \alpha \frac{f(a)+f(b)}{2} \left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a))\right]^{\frac{\alpha}{1-\alpha}} + \beta \frac{g(a)+g(b)}{2} \left[L_{(\frac{1}{\beta}-1)}(g(b), g(a))\right]^{\frac{\beta}{1-\beta}}.$$

*Proof.* The proof directly follows from the proof of Theorem 3.1. □

**Theorem 3.3.** *Let $f_1, f_2, \ldots, f_n$ be* log *$\varphi$-convex functions. Then for $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, we have*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \leq \sum_{i=1}^{n} \left\{\alpha_i \frac{f_i(a)+f_i(b)}{2} \left[L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a))\right]^{\frac{\alpha_i}{1-\alpha_i}}\right\}. \tag{3.1}$$

*Proof.* Since $f_1, f_2, \ldots, f_n$ be log $\varphi$-convex functions and using inequality

$$f_1.f_2\ldots f_n \leq \alpha_1(f_1)^{\frac{1}{\alpha_1}} + \alpha_2(f_2)^{\frac{1}{\alpha_1}} + \ldots + \alpha_n(f_n)^{\frac{1}{\alpha_n}}, \quad \alpha_1, \alpha_2, \ldots, \alpha_n > 0, \sum_{i=1}^{n} \alpha_i = 1,$$

we have

$$\begin{aligned}
&\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \leq \int_0^1 \left\{ \sum_{i=1}^{n} \alpha_i (f_i(a+te^{i\varphi}(b-a)))^{\frac{1}{\alpha_i}} \right\} dt \\
&\leq \int_0^1 \left\{ \sum_{i=1}^{n} \alpha_i [(f_i(a))^{1-t}(f_i(b))^t]^{\frac{1}{\alpha_i}} \right\} dt = \sum_{i=1}^{n} \alpha_i (f_i(a))^{\frac{1}{\alpha_i}} \int_0^1 \left( \frac{f_i(b)}{f_i(a)} \right)^{\frac{t}{\alpha_i}} dt \\
&= \sum_{i=1}^{n} (\alpha_i)^2 (f_i(a))^{\frac{1}{\alpha_i}} \int_0^{\frac{1}{\alpha_i}} \left( \frac{f_i(b)}{f_i(a)} \right)^u du = \sum_{i=1}^{n} (\alpha_i)^2 \frac{(f_i(b))^{\frac{1}{\alpha_i}} - (f_i(a))^{\frac{1}{\alpha_i}}}{\log f_i(b) - \log f_i(a)} \\
&= \sum_{i=1}^{n} (\alpha_i)^2 \frac{(f_i(b))^{\frac{1}{\alpha_i}} - (f_i(a))^{\frac{1}{\alpha_i}}}{f_i(b) - f_i(a)} L(f_i(b), f_i(a)) = \sum_{i=1}^{n} \alpha_i \left[ L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(b), f_i(a)) \\
&\leq \sum_{i=1}^{n} \alpha_i \frac{f_i(a) + f_i(b)}{2} \left[ L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{1-\alpha_i}{\alpha_i}}.
\end{aligned}$$

This completes the proof. □

**Remark 3.4.** *If we suppose, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$ in inequality* (3.1). *Then*

$$\frac{n}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \leq \sum_{i=1}^{n} \left\{ \frac{f_i(a)+f_i(b)}{2} \left[ L_{n-1}(f_i(b), f_i(a)) \right]^{n-1} \right\}.$$

**Theorem 3.5.** *Let $f_1, f_2, \ldots, f_n$ be* log *$\varphi$-concave functions. Then, for $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, we have*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \geq \sum_{i=1}^{n} \left\{ \alpha_i \frac{f_i(a)+f_i(b)}{2} \left[ L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{\alpha_i}{1-\alpha_i}} \right\}.$$

**Theorem 3.6.** *Let $f$ and $g$ be increasing and* log *$\varphi$-convex functions on $I = [a, a+e^{i\varphi}(b-a)]$. Then*

$$\begin{aligned}
&f\left( \frac{2a+e^{i\varphi}(b-a)}{2} \right) L[g(a), g(b)] + g\left( \frac{2a+e^{i\varphi}(b-a)}{2} \right) L[f(a), f(b)] \\
&\leq \frac{1}{b-a} \int_a^b f(x)w(2a+e^{i\varphi}(b-a)-x)dx + L[f(a)g(a), f(b)g(b)].
\end{aligned}$$

*Proof.* Let $f$ and $g$ be log $\varphi$-convex functions. Then we have

$$\begin{aligned}
f(a+te^{i\varphi}(b-a)) &\leq [f(a)]^{1-t}[f(b)]^t \\
g(a+(1-t)e^{i\varphi}(b-a)) &\leq [g(a)]^t[g(b)]^{1-t}.
\end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4 \rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2 < x_3 < x_4$, we have

$$f(a + te^{i\varphi}(b-a))[g(a)]^t[g(b)]^{1-t} + g(a + (1-t)e^{i\varphi}(b-a))[f(a)]^{1-t}[f(b)]^t$$
$$\leq f(a + te^{i\varphi}(b-a))w(a + (1-t)e^{i\varphi}(b-a)) + [f(a)]^{1-t}[f(b)]^t[g(a)]^t[g(b)]^{1-t}.$$

Integrating above inequalities with respect to $t$ on $[0, 1]$, we have

$$\int_0^1 f(a + te^{i\varphi}(b-a))[g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(a + (1-t)e^{i\varphi}(b-a))[f(a)]^{1-t}[f(b)]^t dt$$
$$\leq \int_0^1 f(a + te^{i\varphi}(b-a))g(a + (1-t)e^{i\varphi}(b-a))dt + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^t[g(b)]^{1-t}dt.$$

Now, since $f$ and $g$ are increasing, then, we have

$$\int_0^1 f(a + te^{i\varphi}(b-a))dt \int_0^1 [g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(a + (1-t)e^{i\varphi}(b-a))dt \int_0^1 [f(a)]^{1-t}[f(b)]^t dt$$
$$\leq \int_0^1 f(a + te^{i\varphi}(b-a))g(a + (1-t)e^{i\varphi}(b-a))dt + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^t[g(b)]^{1-t}dt.$$

Now after simple integration, we have

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dxL[g(a), g(b)]$$
$$+ \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} g(2a + e^{i\varphi}(b-a) - x)dxL[f(a), f(b)]$$
$$\leq \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(2a + e^{i\varphi}(b-a) - x)dx + L[f(a)g(a), f(b)g(b)].$$

Now, using the left hand side of Hermite-Hadamard's inequality for $\log \varphi$-convex functions, we have

$$f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)L[g(a), g(b)] + g\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)L[f(a), f(b)]$$
$$\leq \frac{1}{b-a} \int_a^b f(x)w(2a + e^{i\varphi}(b-a) - x)dx + L[f(a)g(a), f(b)g(b)].$$

The desired result. $\square$

**Theorem 3.7.** *Let $f_1, f_2, \dots, f_n$ be differentiable* log *$\varphi$-convex functions on $I^0$(interior of I). Then, we have*

$$\left.\begin{array}{l} \int\limits_a^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \geq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_2(v)f_3(v)\dots f_n(v)\exp\left(\Delta_1\right)dv \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_3(v)\dots f_n(v)\exp\left(\Delta_2\right)dv \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_2(v)\dots f_{n-1}(v)\exp\left(\Delta_n\right)dv \end{array}\right\}, \tag{3.2}$$

*where*

$$\Delta_i = \left\langle \frac{f'_{\varphi_i}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_i(\frac{2a+e^{i\varphi}(b-a)}{2})}, v - \frac{2a+e^{i\varphi}(b-a)}{2} \right\rangle$$

*Proof.* Since $f_1, f_2 \dots, f_n$ be differentiable log $\varphi$-convex functions, so we have

$$f_1(v) \geq f_1(u)\exp\left[\left\langle \frac{f'_{\varphi_1}(u)}{f_1(u)}, v-u \right\rangle\right], \tag{3.3}$$

$$f_2(v) \geq f_2(u)\exp\left[\left\langle \frac{f'_{\varphi_2}(u)}{f_2(u)}, v-u \right\rangle\right], \tag{3.4}$$

$$\vdots$$

$$f_n(v) \geq f_n(u)\exp\left[\left\langle \frac{f'_{\varphi_n}(u)}{f_n(u)}, v-u \right\rangle\right], \tag{3.5}$$

Multiplying (3.3) by $\alpha_1 f_2(v)f_3(v)\dots f_n(v)$, (3.4) by $\alpha_2 f_1(v)f_3(v)\dots f_n(v)$ and (3.5) by $\alpha_n f_1(v)f_2(v)\dots f_{n-1}(v)$ respectively and then adding the resultant, we have

$$\left.\begin{array}{l} \sum\limits_{i=1}^{n} f_i(v) \\ \geq \alpha_1 f_1(u)f_2(v)f_3(v)\dots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_1}(u)}{f_1(u)}, v-u \right\rangle\right] \\ \quad +\alpha_2 f_2(u)f_1(v)f_3(v)\dots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_2}(u)}{f_2(u)}, v-u \right\rangle\right] \\ \quad \vdots \\ \quad +\alpha_n f_n(u)f_1(v)f_2(v)\dots f_{n-1}(v)\exp\left[\left\langle \frac{f'_{\varphi_n}(u)}{f_n(u)}, v-u \right\rangle\right] \end{array}\right\}. \tag{3.6}$$

Putting $u = \frac{2a+e^{i\varphi}(b-a)}{2}$ in (3.6), we have

$$\left.\begin{array}{l} \sum\limits_{i=1}^{n} f_i(v) \\ \geq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) f_2(v) f_3(v)\ldots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_1}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_1(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle\right] \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) f_1(v) f_3(v)\ldots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_2}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_2(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle\right] \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) f_1(v) f_2(v)\ldots f_{n-1}(v)\exp\left[\left\langle \frac{f'_{\varphi_n}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_n(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle\right] \end{array}\right\}.$$

Integrating both sides of above inequality on $[a, a+e^{i\varphi}(b-a)]$, we have

$$\left.\begin{array}{l} \int\limits_{a}^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \geq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_{a}^{a+e^{i\varphi}(b-a)} f_2(v) f_3(v)\ldots f_n(v)\exp\left(\Delta_1\right)dv \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_{a}^{a+e^{i\varphi}(b-a)} f_1(v) f_3(v)\ldots f_n(v)\exp\left(\Delta_2\right)dv \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_{a}^{a+e^{i\varphi}(b-a)} f_1(v) f_2(v)\ldots f_{n-1}(v)\exp\left(\Delta_n\right)dv \end{array}\right\},$$

This completes the proof. □

**Remark 3.8.** *If we suppose,* $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$ *in inequality* (3.2)*. Then*

$$\left.\begin{array}{l} n\int\limits_{a}^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \geq f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_{a}^{a+e^{i\varphi}(b-a)} f_2(v) f_3(v)\ldots f_n(v)\exp\left(\Delta_1\right)dv \\ \quad +f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_{a}^{a+e^{i\varphi}(b-a)} f_1(v) f_3(v)\ldots f_n(v)\exp\left(\Delta_2\right)dv \\ \quad \vdots \\ \quad +f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_{a}^{a+e^{i\varphi}(b-a)} f_1(v) f_2(v)\ldots f_{n-1}(v)\exp\left(\Delta_n\right)dv \end{array}\right\},$$

*where*

$$\Delta_i = \left\langle \frac{f'_{\varphi_i}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_i(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle$$

**Theorem 3.9.** *Let $f_1, f_2, \ldots, f_n$ be differentiable* log *$\varphi$-concave functions on $I^0$(interior of I). Then, we have*

$$\left.\begin{array}{l} \int\limits_a^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \leq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_2(v)f_3(v)\ldots f_n(v)\exp\left(\Delta_1\right)dv \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_3(v)\ldots f_n(v)\exp\left(\Delta_2\right)dv \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_2(v)\ldots f_{n-1}(v)\exp\left(\Delta_n\right)dv \end{array}\right\},$$

*where*

$$\Delta_i = \left\langle \frac{f'_{\varphi_i}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_i(\frac{2a+e^{i\varphi}(b-a)}{2})}, v - \frac{2a+e^{i\varphi}(b-a)}{2} \right\rangle$$

**Remark 3.10.** *If $\varphi = 0$, then* log *$\varphi$-convex (concave) functions become* log*-convex (concave) in classical sense, thus our results continue to hold for* log*-convex (concave) functions.*

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