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Hermite collocation method for approximate solutions of fractional linear differential equations from order $0 < \alpha < 1$

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Abstract. In this paper, approximate solutions of some fractional linear differential equations from order $0 < \alpha < 1$ were obtained by using Hermite collocation method. Three different examples were solved to illustrate the accuracy and efficiency of the method. Obtained results were compared with exact solutions and some results in literature.

1. Introduction

Fractional differential equations have been used to model in dynamical systems, control theory, signal processing, diffusion wave, heat conduction and many areas of sciences [1]. The analytic results on existence and uniqueness of solutions to fractional differential equations have been investigated by some authors [2, 3]. Exact solutions of some fractional differential equations have been investigated by using modification of He's variational iteration method and Laplace transform [4, 5]. The general fractional linear differential equation is given as the following form;

$$a_n(x)D_c^{\alpha_n}y(x) + a_{n-1}(x)D_c^{\alpha_{n-1}}y(x) + \ldots + a_0(x)D_c^{\alpha_0}y(x) = f(x) \tag{1}$$

where $D_c^{\alpha_n}$ is the derivative of y of order α_n in the sense of Caputo fractional differential operator, $y(x)$ is an unknown function of the independent variable x.

Recently, many numerical methods have been used for solving fractional differential equations such as sinc-galerkin method [6], fractional Adams-Bashforth-Moulton method [7], homotopy perturbation method [8], fractional finite difference method [9], adomian decomposition method [10], homotpy analysis method [11], variational iteration method [12], the tau method [13], Legendre operational matrix [14].

The main aim of this paper is to solve the fractional order linear differential equations numerically by using Hermite Collocation Method (HCM). The given method converts the mentioned equations to the linear algebraic systems of which unknowns are Hermite coefficients, by using collocation points. Since expressing this algebraic systems by matrices and using matrix algebra solution of the algebraic system can be obtained easily. Consequently, the numerical solutions of the fractional order linear equations are obtained in terms of truncated Hermite series.

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2. Fractional Calculus

In this section, some important subjects of the fractional calculus which shall be used in this study are given.

2.1. Gamma Function

The gamma function $\Gamma(p)$ is defined by the improrer integral

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx. \tag{2}$$

Gamma function have proporties that

(i) convergent for $p > 0$
(ii) divergent for $p \leq 0$
(iii) $\Gamma(p+1) = p\Gamma(p)$ for $p > 0$
(iv) $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$.

2.2. Beta Function

The beta function $B(p, q)$ is defined by

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad (p \geq 0,\ q \geq 0). \tag{3}$$

Beta function is

(i) convergent for $p > 0$ and $q > 0$
(ii) divergent for $p \leq 0$ and $q \leq 0$.

Beta function is written using the gamma function as follows

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{4}$$

2.3. Caputo Fractional Derivative

The Caputo fractional derivative is defined by

$${}^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \qquad m-1 < \alpha < m \tag{5}$$

for $\alpha \in \mathbb{R}^+$, $m \in \mathbb{N}^+$ and $x \geq 0$.

3. Hermite Polynomials

The Hermite polynomials $H_n(x)$ can be defined by

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}, \ \ n = 0, 1, \ldots. \tag{6}$$

The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain $(-\infty, \infty)$ with weighting function e^{-x^2}.

The first few Hermite polynomials are given as follows

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2-2\\
H_3(x) &= 8x^3-12x\\
H_4(x) &= 16x^4-48x^2+12\\
H_5(x) &= 32x^5-160x^3+120x.
\end{aligned}$$

4. Implementation of the HCM

In this section, the approximate solutions of the truncated Hermite series form can be obtained as follow:

$$y(x)=\sum_{n=0}^{N} a_n H_n(x^\alpha) \tag{7}$$

of fractional linear differential equation

$$\sum_{k=0}^{m} P_k(x) D_c^{k\alpha} y(x) = g(x) \tag{8}$$

with conditions for $\lfloor m.\alpha \rfloor = t \in \mathbb{N}_0,\ a \le x \le b$

$$\sum_{k=0}^{t-1} [a_{jk}(x) D_c^{k\alpha} y(a) + b_{jk}(x) D_c^{k\alpha} y(b)] = \lambda_j, \qquad j=0,1,2,\cdots,t-1. \tag{9}$$

Here a_n are the unknown Hermite coefficients, N can be chosen any positive integer such that $N \ge t,\ 0 < \alpha \le 1$.
To find a solution in (7) of the problem (8) with the conditions (9), the collocation points can be used by

$$x_i = a + \left(\frac{b-a}{N}\right) i,\ \ i=0,1,\cdots,N,\ \ 0<a\le x\le b. \tag{10}$$

The matrix form of the approximate solution $y(x)$ given by (7) can be written as

$$y(x) = H(x^\alpha) A \tag{11}$$

where $H(x^\alpha) = [H_0(x^\alpha)\ H_1(x^\alpha)\ \cdots\ H_N(x^\alpha)]$ and $A = [a_0\ a_1\ \cdots\ a_N]^T$. Hermite polynomials given in (6) according to the odd and even values of N, x^α instead of x can be written as follows matrix form;
if N is odd

$$\underbrace{\begin{bmatrix} H_0(x^\alpha)\\ H_1(x^\alpha)\\ \vdots\\ H_{N-1}(x^\alpha)\\ H_N(x^\alpha)\end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0\\ 0 & 2^1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ (-1)^{\left(\frac{N-5}{2}\right)}\dfrac{2^0}{0!}\dfrac{(N-1)!}{\left(\frac{N-1}{2}\right)!} & 0 & \cdots & 2^{N-1} & 0\\ 0 & (-1)^{\left(\frac{N-1}{2}\right)}\dfrac{2^1}{1!}\dfrac{N!}{\left(\frac{N-1}{2}\right)!} & \cdots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1\\ x^\alpha\\ \vdots\\ x^{\alpha(N-1)}\\ x^{\alpha N}\end{bmatrix}}_{X^T(x^\alpha)},$$

if N is even

$$\underbrace{\begin{bmatrix} H_0(x^\alpha) \\ H_1(x^\alpha) \\ \vdots \\ H_{N-1}(x^\alpha) \\ H_N(x^\alpha) \end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{\left(\frac{N-2}{2}\right)} \frac{2^1}{1!} \frac{(N-1)!}{\left(\frac{N-2}{2}\right)!} & \cdots & 2^{N-1} & 0 \\ (-1)^{\left(\frac{N-4}{2}\right)} \frac{2^0}{0!} \frac{N!}{\left(\frac{N}{2}\right)!} & 0 & \cdots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^\alpha \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^\alpha)}.$$

The given matrix form is briefly expressed as

$$H^T(x^\alpha) = FX^T(x^\alpha) \tag{12}$$

or

$$H(x^\alpha) = X(x^\alpha)F^T. \tag{13}$$

Substitution of equation (13) into (11) yields

$$y(x) = X(x^\alpha)F^T A. \tag{14}$$

Now, the $k\alpha$-th order Caputo fractional derivative of equation (14) is written as

$$D_c^{k\alpha} y(x) = D_c^{k\alpha} X(x^\alpha)F^T A \tag{15}$$

or equivalently:

$$D_c^{k\alpha} X(x^\alpha) = X(x^\alpha)(B^T)^k \tag{16}$$

where the matrix B is defined as follow:

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Gamma(\alpha+1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} & 0 \end{bmatrix}.$$

If, we substitute equation (16) into equation (15) we have:

$$D_c^{k\alpha} y(x) = X(x^\alpha)(B^T)^k F^T A. \tag{17}$$

For the collocation points $x = x_i, \quad i = 0, 1, 2, \cdots, N$, equation (8) is rewritten as follows

$$\sum_{k=0}^{m} P_k(x_i) D_c^{k\alpha} y(x_i) = g(x_i). \tag{18}$$

The matrix form of equation (18) is given as follows

$$\underbrace{\begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}}_{P_k} \underbrace{\begin{bmatrix} D_c^{k\alpha} y(x_0) \\ D_c^{k\alpha} y(x_1) \\ \vdots \\ D_c^{k\alpha} y(x_N) \end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}}_{G}.$$

Therefore, the matrix form is equivalent to

$$\sum_{k=0}^{m} P_k Y^{k\alpha} = G. \tag{19}$$

In equtaion (17), $x = x_i$ is written for obtaining of the matrix $Y^{k\alpha}$, equation (20) is found by

$$D_c^{k\alpha} y(x_i) = X(x_i^\alpha)(B^T)^k F^T A. \tag{20}$$

The matrix form of this equation can be written as follows

$$\underbrace{\begin{bmatrix} D_c^{k\alpha} y(x_0) \\ D_c^{k\alpha} y(x_1) \\ \vdots \\ D_c^{k\alpha} y(x_N) \end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} X(x_0^\alpha) \\ X(x_2^\alpha) \\ \vdots \\ X(x_N^\alpha) \end{bmatrix}}_{X^\alpha} \left[(B^T)^k F^T A\right]. \tag{21}$$

Equation (21) is rewritten as

$$Y^{k\alpha} = X^\alpha (B^T)^k F^T A. \tag{22}$$

Then, by writing equation (22) in equation (19), the following equation is obtained

$$\sum_{k=0}^{m} P_k X^\alpha (B^T)^k F^T A = G. \tag{23}$$

In addition, denoting

$$W = [w_{pq}] = \sum_{k=0}^{m} P_k X^\alpha (B^T)^k F^T, \quad p, q = 0, 1, 2, \cdots, N \tag{24}$$

equation (23) is briefly written by

$$WA = G \text{ or } [W; G] = A. \tag{25}$$

The augmented matrix of equation (25) is written as follows:

$$[W; G] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & g(x_{N-1}) \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(x_N) \end{bmatrix}_{(N+1)x(N+1)}$$

Now, we have to establish the new form of equation (25) under the following initial conditions.

$$D_c^j y(a) = \lambda_j, \quad j = 0, 1, 2, \cdots, t-1. \tag{26}$$

Using equation (20) in condition equation (26), the following equation is found

$$X^\alpha(a)(B^T)^j F^T A = \lambda_j. \tag{27}$$

The matrix form of equation (27) is written as

$$U_j A = \lambda_j \ \text{ or } \ \left[U_j; \lambda_j\right], \ \ j = 0, 1, 2, \cdots, t-1 \tag{28}$$

where

$$U_j = X^\alpha(a)(B^T)^j F^T \equiv \left[u_{j0} \ \ u_{j1} \ \ u_{j2} \ \ \cdots \ \ u_{jN}\right]. \tag{29}$$

After the conditions were added, the formed augmented matrix can be written as follows:

$$[\widetilde{W}; \widetilde{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-t-1)0} & w_{(N-t-1)1} & \cdots & w_{(N-t-1)N} & ; & g(x_{N-t-1}) \\ w_{(N-t)0} & w_{(N-t)1} & \cdots & w_{(N-t)N} & ; & g(x_{N-t}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(t-1)0} & u_{(t-1)1} & \cdots & u_{(t-1)N} & ; & \lambda_{t-1} \end{bmatrix}$$

This augmented matrix is briefly written as

$$\widetilde{W} A = \widetilde{G} \tag{30}$$

If $det(\widetilde{W}) \neq 0$, A the matrix of Hermite coefficients which is solution of equation (30) is found by

$$A = (\widetilde{W})^{-1} \widetilde{G}. \tag{31}$$

Finally, substitution of these coefficients into the truncated Hermite series gives the desired solution of the form:

$$y(x) = \sum_{n=0}^{N} a_n H_n(x^\alpha). \tag{32}$$

5. Numerical Examples and Comparisons

In this section, three different examples of fractional linear differential equations were solved for values 0.3, 0.5, 0.75 of α by using HCM. Obtained results were compared with the exact solutions and some numerical results in literature. All numerical computations were performed by using MatlabR2009b.

5.1. Example 1:

As the first example, the following fractional linear differential equation was solved

$$^{C}D^{\alpha}y(x) + y(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x, \quad x \in [0,1] \tag{33}$$

with the initial condition

$$y(0) = 0 \tag{34}$$

The exact solution was given by $y(x) = x^2 - x$. The approximate solution was obtained by applying Hermite collocation method. In calculations, $N = 7$ and $\alpha = 0.3$ were used. Obtained numerical results were compared with exact solutions in Table 1. In addition, absolute errors were calculated at the selected points of the given interval. Graphs of exact and approximate solution were depicted in Figure 1.

Table 1 : Comparison of HCM solution with exact solution.

x	HCM	Exact Solution	Absolute Error
0.1	-0.08998084	-0.09000000	$1.92x10^{-5}$
0.2	-0.15999265	-0.16000000	$7.35x10^{-6}$
0.3	-0.20999508	-0.21000000	$4.92x10^{-6}$
0.4	-0.23999632	-0.24000000	$3.68x10^{-6}$
0.5	-0.24999710	-0.25000000	$2.90x10^{-6}$
0.6	-0.23999759	-0.24000000	$2.41x10^{-6}$
0.7	-0.20999794	-0.21000000	$2.06x10^{-6}$
0.8	-0.15999824	-0.16000000	$1.76x10^{-6}$
0.9	-0.08999836	-0.09000000	$1.64x10^{-6}$
1	0.00000204	0	$2.04x10^{-6}$

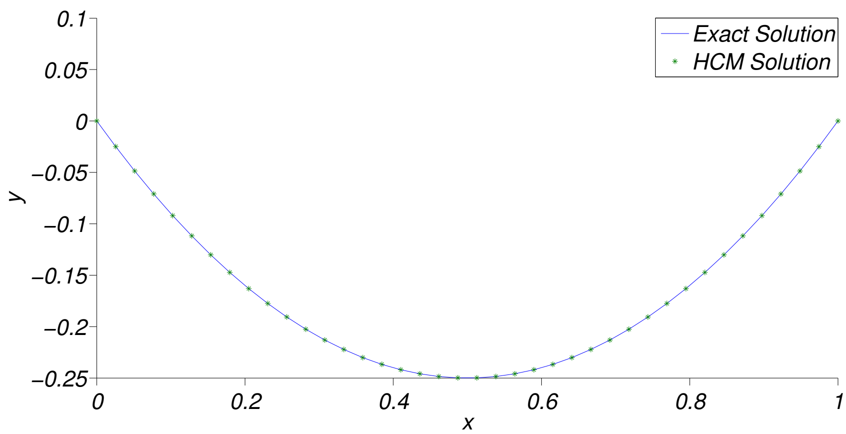

Figure 1: Graphs of exact and approximate solution graphs for Example 1

5.2. Example 2:

As the second example, the following fractional linear differential equation was solved

$$y''(x) - xy'(x) +^{C} D^{0.5}y(x) = -3x^3 - 2x^2 + 6x + 2 + \frac{1}{\Gamma(0.5)}(2.67x^{1.5} + 3.2x^{2.5}) \tag{35}$$

with the nonhomogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 2. \tag{36}$$

The exact solution was given by $y(x) = x^3 + x^2$. First, the non-homogeneous boundary conditions must be homogenized using the transformation $u(x) = y(x) - 2x$. In this case, above fractional linear differential equation with the nonhomogeneous boundary conditions is written as

$$u''(x) - xu'(x) +^C D^{0.5}u(x) = -3x^3 - 2x^2 + 8x + 2 + \frac{1}{\Gamma(0.5)}(2.67x^{1.5} + 3.2x^{2.5} - 4x^{0.5}) \tag{37}$$

with the homogeneous boundary conditions

$$u(0) = 0, \quad u(1) = 0. \tag{38}$$

The approximate solution was obtained by applying Hermite collocation method. In calculations, $N = 6$ and $\alpha = 0.5$ were used. Obtained numerical results were compared with exact solutions in Table 2. In addition, computed absolute errors were compared with absolute errors of Sinc-Galerkin Method [6] at the selected points of the given interval. Graphs of exact and approximate solution were depicted in Figure 2. It is seen that all calculated numerical results are in good agreement with the values of [6] and exact solutions.

Table 2 : Comparison of HCM solution with exact solution.

x	HCM	Exact Solution	Absolute Error	SGM[6]
0.1	0.01162959	0.011	$6.30x10^{-4}$	$5.14x10^{-3}$
0.2	0.04876612	0.048	$7.66x10^{-4}$	$1.97x10^{-3}$
0.3	0.11780338	0.117	$8.03x10^{-4}$	$5.34x10^{-3}$
0.4	0.22478149	0.224	$7.81x10^{-4}$	$3.13x10^{-3}$
0.5	0.37571715	0.375	$7.17x10^{-4}$	$1.40x10^{-4}$
0.6	0.57661971	0.576	$6.20x10^{-4}$	$8.77x10^{-6}$
0.7	0.83349530	0.833	$4.95x10^{-4}$	$2.94x10^{-3}$
0.8	1.15234834	1.152	$3.48x10^{-4}$	$4.64x10^{-3}$
0.9	1.53918230	1.539	$1.82x10^{-4}$	$1.03x10^{-3}$

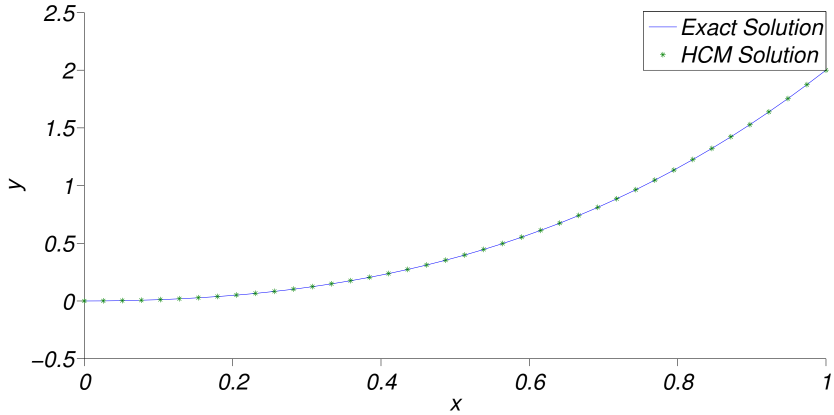

Figure 2: Graphs of exact and approximate solution graphs for Example 2

5.3. Example 3:
As the third example, the following fractional linear differential equation was solved

$$^{C}D^{\alpha}y(x) + y(x) = x^{3+\alpha} + \frac{\Gamma[4+\alpha]}{6}x^3, \quad x \in [0,1] \tag{39}$$

and initial condition

$$y(0) = 0 \tag{40}$$

The exact solution was given by $y(x) = x^{3+\alpha}$. The approximate solution was obtained by applying Hermite collocation method. In calculations, $N = 10$ and $\alpha = 0.75$ were used. Obtained numerical results were compared with exact solutions in Table 3. In addition, computed absolute errors were compared with absolute errors of Fractional Adams-Bashforth-Moulton Method [7] at the selected points of the given interval. Graphs of exact and approximate solution were depicted in Figure 3. It can be said that the presented method gives numerical results with very high accuracy.

Table 3: Comparison of HCM solutions with exact solutions for $\alpha = 0.75$.

x	HCM	Exact Solution	Absolute Error	FABMM[7]
0.0033333	$5.13802767x10^{-10}$	$5.13801414x10^{-10}$	$1.35x10^{-15}$	$3.74x10^{-10}$
0.0066667	$6.91286266x10^{-9}$	$6.91286027x10^{-9}$	$2.39x10^{-15}$	$1.40x10^{-9}$
0.0100000	$3.16227800x10^{-8}$	$3.16227766x10^{-8}$	$3.41x10^{-15}$	$2.97x10^{-9}$
0.0133333	$9.30079952x10^{-8}$	$9.30079907x10^{-8}$	$4.48x10^{-15}$	$5.07x10^{-9}$
0.0166667	$2.14749827x10^{-7}$	$2.14749821x10^{-7}$	$5.35x10^{-15}$	$7.67x10^{-9}$
0.0200000	$4.25364678x10^{-7}$	$4.25364672x10^{-7}$	$6.31x10^{-15}$	$1.08x10^{-8}$
0.0233333	$7.58425539x10^{-7}$	$7.58425532x10^{-7}$	$7.02x10^{-15}$	$1.44x10^{-8}$
0.0266667	$1.25136138x10^{-6}$	$1.25136137x10^{-6}$	$7.88x10^{-15}$	$1.85x10^{-8}$
0.0300000	$1.94627726x10^{-6}$	$1.94627725x10^{-6}$	$8.32x10^{-15}$	$2.31x10^{-8}$
0.0333333	$2.88931769x10^{-6}$	$2.88931768x10^{-6}$	$8.98x10^{-15}$	$2.82x10^{-8}$

6. Conclusion

Hermite collocation method was presented for approximate solutions of fractional linear differential equations from order $0 < \alpha < 1$. The accuracy and efficiency of the method was illustrated by solving three different examples of fractional linear differential equations for values $0.3,\ 0.5,\ 0.75$ of α. According to obtained numerical results, the presented method satisfied very highly acceptable results for the given fractional linear differential equations.

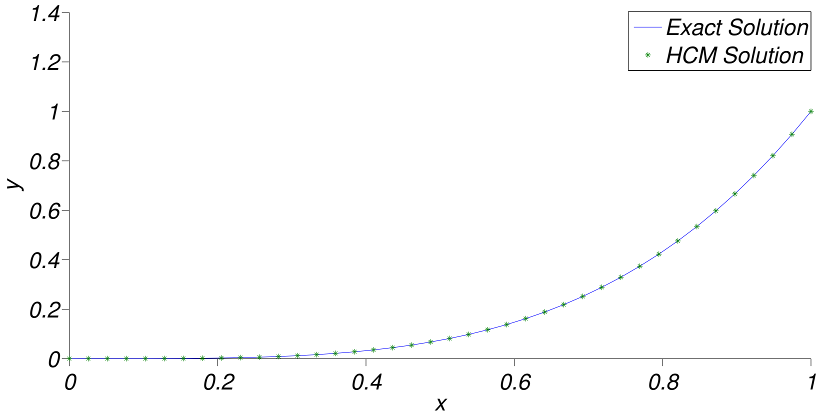

Figure 3: Graphs of exact and approximate solution graphs for Example 3

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