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Tensor approach to finding a basic feasible solution of the transportation problem

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Abstract. A new metric tensor-based approach to transportation problem is used in this paper. We used the North West Corner Rule and Minimum-cost Method with the application of tensor calculus to generalize this problem. As a test example, we analyzed the 3×3 time-dependent transformation of prices with respect to data transfer.

1. Introduction

One of the most common problems that company managers often encounter is the optimization: cost minimization, productivity maximization, resource allocation optimization, etc. In general, it is the problem of finding, under certain constraints, the maximum or minimum value of a function of several variables - objective function. If the objective function as well as the functions describing the constraints, the process of finding the optimal solution is called linear programming (LP). A general approach to formulating a LP problem includes the following standard steps: understanding the problem, identification of decision variables, determining the objective function as a linear combination of decision variables, formulation of the objective function and all constraints as a linear combination of decision variables, and determining both global upper and down limits of decision variables.

The transportation problem (TP), one of the basic problems of transport flow, is a special type of LP problem used to minimize the transportation cost of distributing a single commodity from a number of supply sources to a number of demand destinations obeying the supply limit and demand requirement. Logistics and supply-chain management for reducing cost largely depend on transportation models that can, in the case when cost coefficients and demand quantities are known, lead to efficient algorithms. One of the pioneers of the TP (being a part of transshipment problem) is A. N. Tolstoí [11, 13] who wrote a paper on finding minimal total kilometrage in cargo transportation in space published in a book on transportation planning issued by the National Commissariat of Transportation of the Soviet Union. He presented and explained several approaches for transportation cargo along the railway in the Soviet Union. The first mathematical formulation of TP as it is now known, as well as the corresponding methods, originate from F. L. Hitchcock [4], T. C. Koopmans [7], and in the form of the simplex method, G. B. Dantzig [2]. Later they have been followed by a lot of papers implemented in the fields of industry and business. Although TP is

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quite old, there are still enough scientific works and discussions that deal with specific issues of practical application of analytical methods for solving it [3, 5]. Moreover, apart from the original task, tasks in the field of production, services, management, marketing (optimal placement of machines, auxiliary services, warehouses, selection of the location of services or energy facilities, flow and storage of data, selection of workers) are increasingly being formulated and solved in the form of TP.

The main purpose of this paper is to give a different, more general approach to the transportation problem using an arbitrary metric tensor. In this way, it is possible to consider the influence of more factors on finding the optimal solution to a practical problem than in the standard case.

2. Preliminaries

First, let us define transportation problem (TP).

Suppose that there are m sources, $S_1, S_2, \dots, S_m$, of some goods and n demand destinations, $D_1, D_2, \dots, D_n$. Let a_i represents the capacity of a source i, b_j is demand of the destination j, while c_{ij} correspond to cost of transportation a unit of goods from source S_i to destination D_j. The tabular representation of these data is

	D_1	D_2	$\dots$	D_j	$\dots$	D_n	
S_1	c_{11}	c_{12}	$\dots$	c_{1j}	$\dots$	c_{1n}	a_1
S_2	c_{21}	c_{22}	$\dots$	c_{2j}	$\dots$	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_i	c_{i1}	c_{i2}	$\dots$	c_{ij}	$\dots$	c_{in}	a_i
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_m	c_{m1}	c_{m2}	$\dots$	c_{mj}	$\dots$	c_{mn}	a_m
	b_1	b_2	$\dots$	b_j	$\dots$	b_n	

The goal is to determine the optimal quantities of goods x_{ij} transported from S_i to D_j so that the cost of transportation is minimal.

The mathematical model of TP is [1, 12, 17]

$$\begin{aligned}
&\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
&\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \dots, m, \\
&\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \dots, n, \\
&x_{ij} \geq 0, \quad i = 1, \dots, m, \; j = 1, \dots, n.
\end{aligned} \tag{1}$$

It is known that vectors and matrices, the basic objects in TP, can be viewed as special cases of tensors (tensors of the first and second order). Therefore, it is reasonable to consider TP in the tensor calculus environment.

Riemannian spaces and metric tensors have been studied theoretically [9], but they are applied in different subjects as well [14–16]. In this paper, we will recognize one more application of symmetric metric tensors.

A manifold $\mathcal{M}_N$ equipped with a symmetric metric tensor $\underline{\hat{g}}$ which corresponds to matrix $\left[g_{\underline{ij}}\right]_{N\times N}$, $g_{\underline{ij}} = g_{\underline{ji}}$, is Riemannian space $\mathbb{R}_N$. We assume that matrix of tensor $\hat{g}$ is non-singular, i.e. $\det\left[g_{\underline{ij}}\right] \neq 0$.

In tensor calculus, it has been used the Einstein summation convention with mute indices. For example,

$$a^{i\underline{\alpha}} b_{j\underline{\alpha}} = \sum_{p=1}^{N} a^{ip} b_{jp}, \tag{2}$$

for tensors $\hat{a}$ and $\hat{b}$ of the types $(2,0)$ and $(0,2)$, respectively. The Einstein summation convention applied on tensors of the type $(1,1)$ is equivalent to obtaining separate elements of product of two matrices. For simplicity, let $u = \left[u^i_j\right]$ and $v = \left[v^i_j\right]$ be matrices of the type $N \times N$.

If $w = uv$, then

$$w = \left[w^i_j\right] = \left[\sum_{p=1}^{N} u^i_p v^p_j\right] = \left[u^i_\alpha v^\alpha_j\right].$$

After comparing with the Einstein summation convention, the last expression is presented as $\left[w^i_j\right] = \left[u^i_\alpha v^\alpha_j\right]$, with the mute index α.

Because of non-singularity of the matrix $\left[g_{\underline{ij}}\right]$, the components of metric tensor with upper indices are defined as $\left[g^{\underline{ij}}\right] = \left[g_{\underline{ij}}\right]^{-1}$. According to the Einstein summation convention by mute indices α and because of symmetries $g_{\underline{ij}} = g_{\underline{ji}}$ such as $g^{\underline{ij}} = g^{\underline{ji}}$, the next equalities hold

$$\delta^i_j = g^{\underline{i\alpha}} g_{\underline{j\alpha}} = g^{\underline{i\alpha}} g_{\underline{\alpha j}} = g^{\underline{\alpha i}} g_{\underline{j\alpha}} = g^{\underline{\alpha i}} g_{\underline{\alpha j}}. \tag{3}$$

The symmetric metric tensor with bottom indices has been used for lowering of indices, i.e. if $U^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is a geometrical object of the type (p,q), the lowering of an index i_r and the rising of an index j_s are realized in the next manner

$$U^{i_1 \ldots i_{r-1} \cdot i_{r+1} \ldots i_p}_{i_r j_1 \ldots j_s} = g_{\underline{i_r \alpha}} U^{i_1 \ldots i_{r-1} \alpha i_{r+1} \ldots i_p}_{j_1 \ldots j_q} \quad \text{and} \quad U^{j_s i_1 \ldots i_p}_{j_1 \ldots j_{s-1} \cdot j_{s+1} \ldots j_q} = g^{\underline{j_s \beta}} U^{i_1 \ldots i_p}_{j_1 \ldots j_{s-1} \beta j_{s+1} \ldots j_q}. \tag{4}$$

Finally, because matrix $\left[g^{\underline{ij}}\right]$ is symmetric, it generates an inner product of vectors in next manner:

$$u \cdot_g v = \begin{bmatrix} u_1 & \ldots & u_N \end{bmatrix} \begin{bmatrix} g^{\underline{11}} & \cdots & g^{\underline{1N}} \\ \vdots & \ddots & \vdots \\ g^{\underline{N1}} & \cdots & g^{\underline{NN}} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} = \left[\sum_{p=1}^{N} \sum_{q=1}^{N} u_p g^{\underline{pq}} v_q\right] \equiv \left[u_\alpha g^{\underline{\alpha\beta}} v_\beta\right].$$

According to (4), because the previous product $u_\alpha u_\beta$ with components $g^{\underline{\alpha\beta}}$ of metric tensor $\underline{\hat{g}}$ corresponds to the rising of index α in u_α or index β in v_β, $u_\alpha g^{\underline{\alpha\beta}} = u^\beta$, $g^{\underline{\alpha\beta}} v_\beta = v^\alpha$, we get

$$u_\alpha g^{\underline{\alpha\beta}} v_\beta \equiv u_\alpha v^\alpha \equiv u^\beta v_\beta.$$

In Euclidean space $\mathbb{E}_N$, the components of metric tensor are components of Kronecker delta-symbols with bottom indices, $g_{\underline{ij}} = \delta_{ij}$. Hence, raising and lowering of any index in the tensor $\hat{\tau}$ does not change value of the indexed object. In other words, with respect to the metric of Euclidean space $\mathbb{E}_N$, it is not significant whether the indices in $\tau^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ are upper or lower ones. Components of a vector $\hat{u}$ in Euclidean space may be written as u_i or u^i. Components of matrix w of the type $N \times N$ in space $\mathbb{E}_N$ may be written as w^i_j, w_{ij}, or w^{ij}, equivalently. In an N-dimensional Riemannian space, it is important whether indices are bottom or lowering ones. Positions of indices are important for transformation of indexed objects under transformations of reference systems.

3. Basic feasible solution of transportation problem

Let us recall some terms and facts related to TP (1) [10, 11].

- A matrix $X = \left[x_{ij}\right]_{m \times n}$ of non-negative individual allocations ($x_{ij} \geq 0$) which satisfies the row and column conditions is *feasible solution*.

- A feasible solution such that contains no more than $m+n-1$ non-negative allocations is *basic feasible solution*.
- A basic feasible solution which minimizes the total transportation cost is *optimal solution*.

It is known that TP has an optimal solution if and only if it is balanced, i.e. if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is valid. An unbalanced TP can be made balanced by adding any dummy source if the supply is less than the demand, or dummy destination otherwise, both with zero cost.

The process of solving the transport problem is usually done in two stages: finding of an initial basic feasible solution and checking the optimality with improving the solution until it is optimal. There are a few various methods for each stage. With the intention of giving a new, tensor based approach to the TP, we will indicate two simple methods for determining the initial basic solution, North West Corner Rule and Minimum-cost Method.

In the sequel, we explain these methods followed by an example.

Algorithm 3.1. [1, 6, 8, 17] ***North West Corner Rule Method*** (NWC Method)

Step 1: *Balance the transportation problem if not originally by adding a dummy source or destination making $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, with zero transportation cost in added cells.*
Start with the north west (upper left) corner $(1,1)$, *i.e. put* $i = j = 1$.

Step 2: *Assign the value $x_{ij} = \min\{a_i, b_j\}$ and reduce supply a_i or demand b_j accordingly.*

Step 3: *If a_i is exhausted, go to one cell vertically down, i.e. put $i := i+1$. If b_j is exhausted, go to one cell horizontally right, i.e. put $j := j+1$.*

Step 4: *Repeat Steps 2 and 3 until available quantity is exhausted.*

Algorithm 3.2. [1, 6, 8, 17] ***Minimum-cost Method*** (MC Method)

Step 1: *Identify the cell (i, j) with the smallest cost c_{ij} and allocate $x_{ij} = \min\{a_i, b_j\}$.*

Step 2: *If $\min\{a_i, b_j\} = a_i$, then cross out the ith row and decrease $b_j j$ by a_i. If $\min\{a_i, b_j\} = b_j$, then cross out the jth column and decrease $a_i j$ by b_j. If $\min\{a_i, b_j\} = a_i = b_j$, cross out only one of ith row or jth column.*

Step 3: *Repeat Steps 1 and 2 with uncrossed-out cells until available quantity is exhausted.*

The example of working of the previous algorithms is the following one.

Example 3.3. *There are three routs A, B, and C for data transfer and three auxiliary storage facilities P, Q, and R who keep data before transferring it to the Federal cluster at the end of the day. The capacity of data on routes A, B, and C are* 80, 70, *and* 60 *thousand of terabyte (TB) respectively, and daily capacity of storages* 50, 110, *and* 50 *thousand TB respectively. The cost of transferring one TB of data from each route differs from each route to each storage, according to distance, and can be represented as follows: route A:* 16, 18 *and* 21*; route B:* 17, 19, *and* 14*; route C:* 32, 11, *and* 15 *all to storages P, Q, and R respectively. The problem is how many thousands of TB is to be sent from each route to find a basic feasible solution of the total cost of transfer.*

Representing constraints in Table and applying Algorithm 1, we obtain

$$x_{11} = \min\{80, 50\} = 50; \quad x_{12} = \min\{80-50, 110\} = 30;$$
$$x_{22} = \min\{70, 110-30\} = 70; \quad x_{32} = \min\{60, 110-30-70\} = 10; \quad x_{33} = 50,$$

	P	*Q*	*R*	
A	$\boxed{50}$ 16	$\boxed{30}$ 18	21	80
B	17	$\boxed{70}$ 19	14	70
C	32	$\boxed{10}$ 11	$\boxed{50}$ 15	60
	50	110	50	210

The obtained basic feasible solution and corresponding total cost are

$$X = \begin{bmatrix} 50 & 30 & 0 \\ 0 & 70 & 0 \\ 0 & 10 & 50 \end{bmatrix}, \quad Z = 16 \cdot 50 + 18 \cdot 30 + 19 \cdot 70 + 11 \cdot 10 + 15 \cdot 50 = 3530.$$

The minimum-cost method finds a better starting solution by concentrating on the cheapest routes. The solution is given by

$$\begin{aligned}
&\min\{c_{ij},\ i,j = 1,2,3\} = 11 = c_{32}, \quad x_{32} = \min\{60, 110\} = 60;\\
&\min\{c_{ij},\ i = 1,2, j = 1,2,3\} = 14 = c_{23}, \quad x_{23} = \min\{50, 70\} = 50;\\
&\min\{c_{ij},\ i,j = 1,2\} = 16 = c_{11}, \quad x_{11} = \min\{50, 80\} = 50;\\
&\min\{c_{i2},\ i = 1,2\} = 18 = c_{12}, \quad x_{12} = \min\{110 - 60, 80 - 50\} = 30; \qquad x_{22} = 20.
\end{aligned}$$

	P	*Q*	*R*	
A	$\boxed{50}$ 16	$\boxed{30}$ 18	21	80
B	17	$\boxed{20}$ 19	$\boxed{50}$ 14	70
C	32	$\boxed{60}$ 11	15	60
	50	110	50	210

$$X = \begin{bmatrix} 50 & 30 & 0 \\ 0 & 20 & 50 \\ 0 & 60 & 0 \end{bmatrix}, \quad Z = 16 \cdot 50 + 18 \cdot 30 + 19 \cdot 20 + 14 \cdot 50 + 11 \cdot 60 = 3080. \tag{5}$$

4. Transportation problem and metric tensor

Let us TP (1) present in tensor notation. Suppose that $m = n = N$. If it is not satisfied, we can add some dummy sources or destinations with zero components.

Vectors of supplies and demands are tensors of the first order in the Euclidean space $\mathbb{E}_N$:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}^T = [a_i]^T, \quad \begin{bmatrix} b_1 & b_2 & \cdots & b_N \end{bmatrix} = [b_j], \qquad i, j \in \{1, 2, \ldots, N\}.$$

The cost matrix is a tensor of the second order

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ & \cdots & & \\ c_{N1} & c_{N2} & \ldots & c_{NN} \end{bmatrix} = [c_{ij}].$$

Decision variables are

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ & \cdots & & \\ x_{N1} & x_{N2} & \ldots & x_{NN} \end{bmatrix} = [x_{ij}].$$

In that manner, TP (1), with respect to the Einstein summation convention as in (2), is equivalent to

$$\begin{aligned}
\text{Minimize } Z &= \delta^{\alpha\gamma}\delta^{\beta\delta}c_{\alpha\beta}x_{\gamma\delta},\\
\delta^{\alpha\beta}e_\alpha x_{i\beta} &= a_i,\\
\delta^{\alpha\beta}e_\alpha x_{\beta j} &= b_j,\\
x_{ij} &\geq 0,\\
i, j &= 1, \ldots, N,
\end{aligned} \tag{6}$$

where δ^{ij} is Kronecker's delta symbol with upper indices, and $e_i = 1$.

As we have already mentioned, δ_{ij} are components of metric tensor of the Euclidean space $\mathbb{E}_N$. For this reason, the tensor form of objective function Z given in (6) may be equivalently expressed as

$$Z = c_{\alpha\beta}x^{\alpha\beta} \quad \text{or} \quad Z = c^{\alpha\beta}x_{\alpha\beta}. \tag{7}$$

If x^{ij} is solution of transporting problem (7), the corresponding solution x_{ij} with bottom indices is obtained in the manner (4) as

$$x_{ij} = \delta_{i\alpha}\delta_{j\beta}x^{\alpha\beta}. \tag{8}$$

We will generalize the transportation problem (6) with respect to $m = n = N$ below. Instead of Kronecker delta symbols with upper indices, we will use the symmetric metric tensor with upper indices. In that way, the objective function becomes more general and permits impact of some new parameters.

In the general transportation problem with respect to symmetric metric tensor $\underline{\hat{g}}$, the objective function $\mathcal{Z}$ is expressed as

$$\mathcal{Z} = g^{\underline{\alpha\gamma}}g^{\underline{\beta\delta}}c_{\alpha\beta}x_{\gamma\delta}.$$

The components of corresponding matrix of costs are

$$C^{ij} = g^{\underline{i\gamma}}g^{\underline{j\delta}}c_{\underline{\gamma\delta}}. \tag{9}$$

The solution of this problem is presented in following theorem.

Theorem 4.1. *Let $\left[c_{ij}\right]$ be a constant matrix and let $\left[C^{ij}\right]$, for $C^{ij} = g^{\underline{i\alpha}}g^{\underline{j\beta}}c_{\alpha\beta}$ be the matrix of cost. The basic feasible solution of transportation problem*

$$\begin{aligned} \textit{Minimize } \mathcal{Z} = g^{\underline{\alpha\gamma}}g^{\underline{\beta\delta}}c_{\alpha\beta}x_{\gamma\delta} \\ \delta^{\alpha\beta}e_\alpha x_{i\beta} = a_i \\ \delta^{\alpha\beta}e_\alpha x_{\beta j} = b_j \\ x_{ij} \geq 0 \\ i, j = 1, \ldots, N \end{aligned} \tag{10}$$

is

$$x_{ij} = g_{\alpha i}g_{\beta j}x^{\alpha\beta}, \tag{11}$$

where $x^{\alpha\beta}$ is the basic feasible solution of transportation problem

$$\begin{aligned} \textit{Minimize } Z = \delta^{\alpha\gamma}\delta^{\beta\delta}c_{\alpha\beta}x_{\gamma\delta}, \\ \delta^{\alpha\beta}e_\alpha x_{i\beta} = a_i, \\ \delta^{\alpha\beta}e_\alpha x_{\beta j} = b_j, \\ x_{ij} \geq 0, \\ i, j = 1, \ldots, N. \end{aligned} \tag{12}$$

If $\mathcal{G} = \left[g_{\underline{ij}}\right]$ is matrix consisted of components of metric tensor and if $\mathcal{X} = \left[x^{ij}\right]$ is matrix consisted of solutions $x^{ij} = \delta^{i\alpha}\delta^{j\beta}x_{\alpha\beta}$ of problem (12)*, the matrix of solutions of problem* (10) *is*

$$\left[x_{ij}\right] = \mathcal{G}\mathcal{X}\mathcal{G}. \tag{13}$$

Proof. The transportation problem (10) is equivalent to the problem

$$\mathcal{Z} = c_{\alpha\beta}x^{\alpha\beta},$$

where indices in $x^{\alpha\beta}$ are raised with respect to nonconstant metric tensor $\hat{g}$.

After applying some of the TP solving methods, we obtain the corresponding components of $x^{\alpha\beta}$. The solution of problem (10) is obtained after lowering indices from $x^{\alpha\beta}$.

Because left indices in elements of matrices m_{ij} denote the row, and the right ones denote the column where the element is placed, the equation (13) holds directly from tensor expression of this solution. □

Example 4.2. *Let us consider problem presented in Example 1 with additional information on data transfer reliability. Let $g_{\underline{ij}}$ denote special benefits if facility P, Q, R use routs A, B, C respectively.*

In the case of $g^{\underline{ij}} = \delta^{ij}$, the next equality holds

$$Z = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} x_{ij}.$$

With respect to a symmetric metric tensor $\hat{g}$, whose components are $g_{\underline{ij}}$, one obtains the following equation

$$\mathcal{Z} = \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} x_{ij}$$

for

$$\begin{aligned}
d_{11} &= c_{11}(g_{\underline{11}})^2 + c_{12}g_{\underline{11}}g_{\underline{12}} + c_{21}g_{\underline{11}}g_{\underline{12}} + c_{22}(g_{\underline{12}})^2 \\
&\quad + c_{13}g_{\underline{11}}g_{\underline{13}} + c_{31}g_{\underline{11}}g_{\underline{13}} + c_{23}g_{\underline{12}}g_{\underline{13}} + c_{32}g_{\underline{12}}g_{\underline{13}} + c_{33}(g_{\underline{13}})^2, \\
d_{12} &= c_{11}g_{\underline{11}}g_{\underline{12}} + c_{21}(g_{\underline{12}})^2 + c_{31}g_{\underline{12}}g_{\underline{13}} + c_{\underline{12}}g_{\underline{11}}g_{\underline{22}} \\
&\quad + c_{22}g_{\underline{12}}g_{\underline{22}} + c_{32}g_{\underline{13}}g_{\underline{22}} + c_{13}g_{\underline{11}}g_{\underline{23}} + c_{23}g_{\underline{12}}g_{\underline{23}} + c_{33}g_{\underline{13}}g_{\underline{23}}, \\
d_{13} &= c_{11}g_{\underline{11}}g_{\underline{13}} + c_{21}g_{\underline{12}}g_{\underline{13}} + c_{31}(g_{\underline{13}})^2 + c_{12}g_{\underline{11}}g_{\underline{23}} \\
&\quad + c_{22}g_{\underline{12}}g_{\underline{23}} + c_{32}g_{\underline{13}}g_{\underline{23}} + c_{13}g_{\underline{11}}g_{\underline{33}} + c_{23}g_{\underline{12}}g_{\underline{33}} + c_{33}g_{\underline{13}}g_{\underline{33}}, \\
d_{21} &= c_{11}g_{\underline{11}}g_{\underline{12}} + c_{12}(g_{\underline{12}})^2 + c_{13}g_{\underline{12}}g_{\underline{13}} + c_{\underline{21}}g_{\underline{11}}g_{\underline{22}} \\
&\quad + c_{22}g_{\underline{12}}g_{\underline{22}} + c_{23}g_{\underline{13}}g_{\underline{22}} + c_{31}g_{\underline{11}}g_{\underline{23}} + c_{32}g_{\underline{12}}g_{\underline{23}} + c_{33}g_{\underline{13}}g_{\underline{23}}, \\
d_{22} &= c_{11}(g_{\underline{12}})^2 + c_{12}g_{\underline{12}}g_{\underline{22}} + c_{21}g_{\underline{12}}g_{\underline{22}} + c_{22}(g_{\underline{22}})^2 \\
&\quad + c_{13}g_{\underline{12}}g_{\underline{23}} + c_{31}g_{\underline{12}}g_{\underline{23}} + c_{23}g_{\underline{22}}g_{\underline{23}} + c_{32}g_{\underline{22}}g_{\underline{23}} + c_{33}(g_{\underline{23}})^2, \\
d_{23} &= c_{11}g_{\underline{12}}g_{\underline{13}} + c_{21}g_{\underline{13}}g_{\underline{22}} + c_{12}g_{\underline{12}}g_{\underline{23}} + c_{31}g_{\underline{13}}g_{\underline{23}} \\
&\quad + c_{22}g_{\underline{22}}g_{\underline{23}} + c_{32}(g_{\underline{23}})^2 + c_{13}g_{\underline{12}}g_{\underline{33}} + c_{23}g_{\underline{22}}g_{\underline{33}} + c_{33}g_{\underline{23}}g_{\underline{33}}, \\
d_{31} &= c_{11}g_{\underline{11}}g_{\underline{13}} + c_{12}g_{\underline{12}}g_{\underline{13}} + c_{13}(g_{\underline{13}})^2 + c_{21}g_{\underline{11}}g_{\underline{23}} \\
&\quad + c_{22}g_{\underline{12}}g_{\underline{23}} + c_{23}g_{\underline{13}}g_{\underline{23}} + c_{31}g_{\underline{11}}g_{\underline{33}} + c_{32}g_{\underline{12}}g_{\underline{33}} + c_{33}g_{\underline{13}}g_{\underline{33}}, \\
d_{32} &= c_{11}g_{\underline{12}}g_{\underline{13}} + c_{12}g_{\underline{13}}g_{\underline{22}} + c_{21}g_{\underline{12}}g_{\underline{23}} + c_{13}g_{\underline{13}}g_{\underline{23}} \\
&\quad + c_{22}g_{\underline{22}}g_{\underline{23}} + c_{23}(g_{\underline{23}})^2 + c_{31}g_{\underline{12}}g_{\underline{33}} + c_{32}g_{\underline{22}}g_{\underline{33}} + c_{33}g_{\underline{23}}g_{\underline{33}}, \\
d_{33} &= c_{11}(g_{\underline{13}})^2 + c_{12}g_{\underline{13}}g_{\underline{23}} + c_{22}(g_{\underline{23}})^2 \\
&\quad + c_{13}g_{\underline{13}}g_{\underline{33}} + c_{31}g_{\underline{13}}g_{\underline{33}} + c_{23}g_{\underline{23}}g_{\underline{33}} + c_{32}g_{\underline{23}}g_{\underline{33}} + c_{33}(g_{\underline{33}})^2.
\end{aligned} \tag{14}$$

Suppose that he metric tensor $\underline{\hat{g}}$ corresponds to matrix $[g_{ij}] = \mathrm{diag}\{K, L, M\}$ and $[g^{ij}] = [g_{ij}]^{-1} = \mathrm{diag}\{K^{-1}, L^{-1}, M^{-1}\}$. The transformed matrix of cost is

$$C = C^{ij} = g^{\underline{i\alpha}} g^{\underline{j\beta}} c_{\alpha\beta} = \begin{bmatrix} 16K^{-2} & 18(KL)^{-1} & 21(KM)^{-1} \\ 17(KL)^{-1} & 19L^{-2} & 14(LM)^{-1} \\ 32(KM)^{-1} & 11(LM)^{-1} & 15M^{-2} \end{bmatrix}.$$

Solving transformed problem we have the solution that includes additional parameters. For example, for $K = 2, L = 10, M = 5$, we have

$$C = \begin{bmatrix} 4 & 0.9 & 2.1 \\ 0.85 & 0.19 & 0.28 \\ 3.2 & 0.22 & 0.6 \end{bmatrix}.$$

Applying MC method shown in Algorithm 3.2 we obtain basic feasible solution and corresponding total cost

$$X = \begin{bmatrix} 50 & 0 & 30 \\ 0 & 70 & 0 \\ 0 & 40 & 20 \end{bmatrix}, \quad Z = 16 \cdot 50 + 21 \cdot 30 + 19 \cdot 70 + 11 \cdot 40 + 15 \cdot 20 = 3500.$$

Note that total cost is worse then one obtained by MC method applied on non-transformed problem (5), but here choice of solution includes offered benefits.

4.1. The generalized feasible solution of a Transportation problem

There are three routs A, B, and C for data transfer and three auxiliary storage facilities P, Q, and R who keep data before transferring it to the Federal cluster at the end of the day. The capacity of data on routes A, B, and C are 80, 70, and 60 thousand of terabyte (TB) respectively, and daily capacity of storages 50, 90, and 70 thousand TR respectively. The cost of transferring one TB of data from each route differs from each route to each storage, according to distance, and can be represented as follows: route A: $A_1(t)$, $A_2(t)$ and $A_3(t)$; route B: $B_1(t)$, $B_2(t)$, and $B_3(t)$; route C: $C_1(t)$, $C_2(t)$, and $C_3(t)$ all to storages P, Q, and R respectively. The problem is how many thousands of TB is to be sent in one hour interval ($0 \le t \le 60$) from each route to minimize the total cost of transfer.

The costs functions and constraints are presented in Table

	P	Q	R	
A	$\boxed{x_{11}(t)}$ $A_1(t)$	$\boxed{x_{12}(t)}$ $A_2(t)$	$\boxed{x_{13}(t)}$ $A_3(t)$	$a_1(t)$
B	$\boxed{x_{21}(t)}$ $B_1(t)$	$\boxed{x_{22}(t)}$ $B_2(t)$	$\boxed{x_{23}(t)}$ $B_3(t)$	$a_2(t)$
C	$\boxed{x_{31}(t)}$ $C_1(t)$	$\boxed{x_{32}(t)}$ $C_2(t)$	$\boxed{x_{33}(t)}$ $C_3(t)$	$a_3(t)$
	$b_1(t)$	$b_2(t)$	$b_3(t)$	

(15)

for $b_3(t) = a_1(t) + a_2(t) + a_3(t) - b_1(t) - b_2(t)$.

Applying NWC ethod, we get the values of a basic feasible solution $[x_{ij}]$.

With respect to $a_1(t) = 80$, $a_2(t) = 70$, $a_3(t) = 60$, $b_1(t) = 50$, $b_2(t) = 90$, $b_3(t) = 70$, and the functions $A_i = A_i(t)$, $B_i = B_i(t)$, $C_i = C_i(t)$, $i = 1, 2, 3$,

$$A_1 = \begin{cases} -\frac{2}{3}t + 50, & t \in [0, 45) \\ 20, & t \in [45, 55) \\ 6t - 310, & t \in [55, 60], \end{cases} \quad A_2 = \begin{cases} -\frac{7}{45}t + 70, & t \in [0, 45), \\ \frac{4}{3}t - 50, & t \in [45, 60] \end{cases} \quad A_3 = -\frac{1}{6}t + 60, \quad t \in [0, 60],$$

$$B_1 = \begin{cases} -\frac{4}{5}t + 80, & t \in [0, 50) \\ t - 10, & t \in [50, 60] \end{cases} \quad B_2 = \begin{cases} -\frac{8}{9}t + 60, & t \in [0, 45) \\ t - 25, & t \in [45, 55) \\ 4t - 190, & t \in [55, 60] \end{cases} \quad B_3 = -\frac{3}{4}t + 50, \quad t \in [0, 60]$$

$$C_1 = -t + 80, \quad t \in [0, 60], \quad C_2 = \begin{cases} -\frac{6}{5}t + 70, & t \in [0, 50), \\ 3t - 140, & t \in [50, 60], \end{cases} \quad C_3 = \begin{cases} -\frac{4}{9}t + 60, & t \in [0, 45) \\ -2t + 130, & t \in [45, 55) \\ -2t + 140, & t \in [55, 60]. \end{cases}$$

we obtain the next basic fesible solution

$$X = \begin{bmatrix} 50 & 30 & 0 \\ 0 & 60 & 10 \\ 0 & 0 & 60 \end{bmatrix}, \quad Z(t) = 50 \cdot A_1(t) + 30 \cdot A_2(t) + 60 \cdot B_2(t) + 10 \cdot B_3(t) + 60 \cdot C_3(t).$$

The total cost function is

$$Z(t) = \begin{cases} -\dfrac{251}{2}t + 12300, & t \in [0, 45) \\ -\dfrac{55}{2}t + 6300, & t \in [45, 55) \\ \dfrac{905}{2}t - 19500, & t \in [55, 60]. \end{cases} \tag{16}$$

The graphical representation of this function is

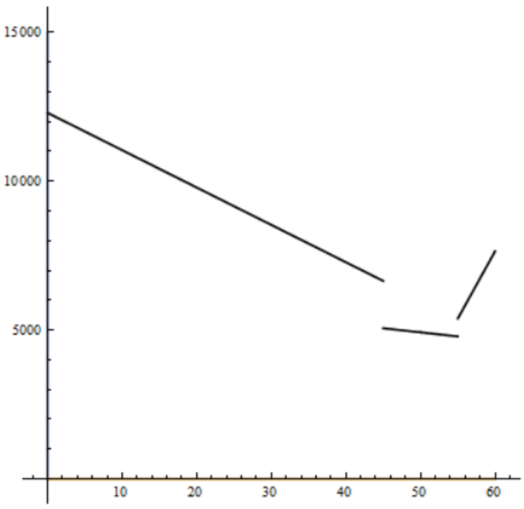

Figure 1: Graphical representation of the function $Z(t)$

From Figure 1 one can see that function $Z(t)$, representing information flow, is decreasing for $t \in [0, 45]$, also decreasing for $t \in [45, 55]$, and at the last part, where $t \in [55, 60]$ function Z(t) in increasing.

With respect to metric tensor $\hat{g}$, whose components are expressed as the following matrix

$$\left[g_{\underline{ij}}\right] = \begin{bmatrix} f(t) & 0 & 0 \\ 0 & f^2(t) & 0 \\ 0 & 0 & f^3(t) \end{bmatrix}, \tag{17}$$

for a function f, $f^k(t) = \underbrace{f(t) \cdot \ldots \cdot f(t)}_{k \text{ times}}$, we obtain transformed cost function

$$\mathcal{Z}(t) = \begin{cases} \mathcal{Z}_1(t), & 0 \le x < 45 \\ \mathcal{Z}_2(t), & 45 \le x < 50 \\ \mathcal{Z}_3(t), & 50 \le x < 55 \\ \mathcal{Z}_4(t), & 55 \le x \le 60, \end{cases} \tag{18}$$

where

$$\begin{aligned}
\mathcal{Z}_1(t) &= 50\cdot\left(-\frac{2}{3}t+50\right)\cdot f^2(t)+30\cdot\left(-\frac{7}{45}t+70\right)\cdot f^3(t)+60\cdot\left(-\frac{4}{5}t+80\right)\cdot f^3(t)\\
&+10\cdot\left(-\frac{8}{9}t+60\right)\cdot f^4(t)+60\cdot\left(-\frac{4}{9}t+60\right)\cdot f^6(t),\\
\mathcal{Z}_2(t) &= 50\cdot 20\cdot f^2(t)+30\cdot\left(\frac{4}{3}t-50\right)\cdot f^3(t)+60\cdot\left(-\frac{4}{5}t+80\right)\cdot f^3(t)\\
&+10\cdot(t-25)\cdot f^4(t)+60\cdot(-2t+130)\cdot f^6(t),\\
\mathcal{Z}_3(t) &= 50\cdot 20\cdot f^2(t)+30\cdot\left(\frac{4}{3}t-50\right)\cdot f^3(t)+60\cdot(t-10)\cdot f^3(t)\\
&+10\cdot(t-25)\cdot f^4(t)+60\cdot(-2t+130)\cdot f^6(t),\\
\mathcal{Z}_4(t) &= 50\cdot(6t-310)\cdot f^2(t)+30\cdot\left(\frac{4}{3}t-50\right)\cdot f^3(t)+60\cdot(t-10)\cdot f^3(t)\\
&+10\cdot(4t-190)\cdot f^4(t)+60\cdot(-2t+140)\cdot f^6(t).
\end{aligned}$$

With respect to the metric tensor $\underline{\hat{g}}$ given by (17), one gets

$$\begin{aligned}
&\mathcal{G}^{-1}=[g^{\underline{ij}}]:\qquad g^{\underline{11}}=\frac{1}{f(t)},\quad g^{\underline{22}}=\frac{1}{f^2(t)},\quad g^{\underline{33}}=\frac{1}{f^3(t)},\quad g^{\underline{ij}}=0,\ i\neq j,\\
&\mathcal{Z}=g^{\underline{\alpha\gamma}}g^{\underline{\beta\delta}}c_{\alpha\beta}x_{\gamma\delta}\\
&\quad=g^{\underline{11}}g^{\underline{11}}c_{11}x_{11}+g^{\underline{11}}g^{\underline{22}}c_{12}x_{12}+g^{\underline{11}}g^{\underline{33}}c_{13}x_{13}+g^{\underline{22}}g^{\underline{11}}c_{21}x_{21}+g^{\underline{22}}g^{\underline{22}}c_{22}x_{22}\\
&\quad+g^{\underline{22}}g^{\underline{33}}c_{23}x_{23}+g^{\underline{33}}g^{\underline{11}}c_{31}x_{31}+g^{\underline{33}}g^{\underline{22}}c_{32}x_{32}+g^{\underline{33}}g^{\underline{33}}c_{33}x_{33}\\
&\quad=\frac{50A_1(t)}{f^2(t)}+\frac{30A_2(t)}{f^3(t)}+\frac{60B_2(t)}{f^4(t)}+\frac{10C_2(t)}{f^5(t)}+\frac{60C_3(t)}{f^6(t)},
\end{aligned}$$

such as

$$X=[x_{ij}]=\begin{bmatrix}50&30&0\\0&60&0\\0&10&60\end{bmatrix},\quad \mathcal{X}=[x^{ij}]=\mathcal{G}^{-1}X\mathcal{G}^{-1}=\begin{bmatrix}\frac{50}{f^2(t)}&\frac{30}{f^3(t)}&0\\0&\frac{60}{f^4(t)}&0\\0&\frac{10}{f^5(t)}&\frac{60}{f^6(t)}\end{bmatrix}.$$

5. Conclusion

Matrix of costs used in the transportation problem (6) is constant. We generalized this approach by defying the matrix of costs C given by (9). The matrix C is matrix-valued function. For this reason, it generalizes the starting approach (1) with respect to different variables. Also, the tensor-based feasible solution to transportation problem related to data transfer was presented. The standard transportation problem (1) is related to inner product with unit matrix (Euclidean space). In this paper, using symmetric matrix (metric tensor in the Riemannian space) instead of unit matrix, generalizes the TP approach. This generalization enables consideration of TP with a external factors affection the goal function.

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