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Remark on general sum-connectivity index

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Abstract. Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple connected graph with n vertices and m edges with vertex degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n > 0$. If ith and jth vertices are adjacent, it is denoted as $i \sim j$. Topological degree-based index of graph $H_\alpha = \sum_{i \sim j} (d_i + d_j)^\alpha$, where α is an arbitrary real number, is referred to as general sum-connectivity index. In this paper we prove inequality that connects invariants H_α, $H_{\alpha-1}$ and $H_{\alpha-2}$. Using that inequality, in some special cases we obtain lower bounds for some other graph invariants.

1. Introduction

Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple connected graph with n vertices and m edges. Denote by $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, and $d(e_1) \ge d(e_2) \ge \cdots \ge d(e_m)$, sequences of vertex and edge degrees, respectively. Throughout this paper we use standard notation: $\Delta_e = d(e_1) + 2$, $\Delta_{e_2} = d(e_2) + 2$, and $\delta_e = d(e_m) + 2$. If two vertices are adjacent we denote it as $i \sim j$. As usual, $L(G)$ denotes a line graph.

In [9] Gutman and Trinajstić defined vertex-degree-based topological indices, named the first and the second Zagreb indices M_1 and M_2, as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i \sim j} d_i d_j.$$

It is noticed (see [3]) that the first Zagreb index can be also expressed as

$$M_1 = \sum_{i \sim j} (d_i + d_j). \tag{1}$$

Details on the first Zagreb index and its applications can be found in [1, 2, 8, 10, 11].

In [7], in analogy to the first Zagreb index, the vertex-degree-based topological index F was defined as

$$F = F(G) = \sum_{i=1}^{n} d_i^3.$$

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For historical reasons [8] it was named forgotten topological index and can be expressed as

$$F = \sum_{i \sim j} (d_i^2 + d_j^2). \tag{2}$$

A further degree-based graph invariant was defined in [17] and named general sum-connectivity index, H_α, as

$$H_\alpha = H_\alpha(G) = \sum_{i \sim j} (d_i + d_j)^\alpha, \tag{3}$$

where α is an arbitrary real number.

It can easily be seen that $H_1 = M_1$ and $H_2 = F + 2M_2$. We are also interested in another topological index, named harmonic index $H = 2H_{-1}$.

In this paper we prove some inequalities for invariants H_α, $H_{\alpha-1}$ and $H_{\alpha-2}$. In special cases we obtain lower bounds for topological indices F and M_1.

2. Preliminaries

In this section we recall some inequalities for topological indices M_1, M_2, F and H that will be needed for our work.

In [7] the following two inequalities for topological index F were proved

$$F \geq \frac{M_1^2}{2m}, \tag{4}$$

and

$$F \geq \frac{M_1^2}{m} - 2M_2. \tag{5}$$

Equality in (4) holds if and only if G is a regular graph, and in (5) if and only if $L(G)$ is regular graph. We mention that inequality (5) was proved in [6] too.

In [16] and [13] the inequality for graph invariants M_1 and H was proved

$$HM_1 \geq 2m^2, \tag{6}$$

with equality if and only if $L(G)$ is regular graph.

3. Main result

The following theorem establishes the nonlinear relation between invariants H_α, $H_{\alpha-1}$ and $H_{\alpha-2}$.

Theorem 1. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then, for any real α*

$$H_{\alpha-2} H_\alpha \geq H_{\alpha-1}^2 + \Delta_{e_2}^{\alpha-2} \Delta_e^{\alpha-2} (\Delta_e - \Delta_{e_2})^2. \tag{7}$$

Equality holds if and only if $L(G)$ is regular graph.

Proof Let $p = (p_i)$, $i = 1, 2, \ldots, m$, be positive real number sequence, and $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, m$, sequences of non-negative real numbers of similar monotonicity. Then (see [14, 15])

$$T_m(a, b; p) \geq T_{m-1}(a, b; p), \quad m \geq 2, \tag{8}$$

where

$$T_m(a,b;p) = \sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i a_i b_i - \sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} p_i b_i.$$

From (8) we have that $T_m(a,b;p) \geq T_2(a,b;p)$, i.e.

$$\sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i a_i b_i - \sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} p_i b_i \geq p_1 p_2 (a_1 - a_2)(b_1 - b_2). \tag{9}$$

For $p_i = (d(e_i)+2)^{\alpha-2}$, $a_i = b_i = d(e_i)+2$, $i = 1, 2, \ldots, m$, inequality (9) becomes

$$\begin{aligned} &\sum_{i=1}^{m} (d(e_i)+2)^{\alpha-2} \sum_{i=1}^{m} (d(e_i)+2)^{\alpha} - \left(\sum_{i=1}^{m} (d(e_i)+2)^{\alpha-1} \right)^2 \\ &\geq \Delta_{e_2}^{\alpha-2} \Delta_e^{\alpha-2} (\Delta_e - \Delta_{e_2})^2. \end{aligned} \tag{10}$$

It easily can be seen that the following is valid

$$H_\alpha = \sum_{i \sim j} (d_i + d_j)^\alpha = \sum_{i=1}^{m} (d(e_i)+2)^\alpha. \tag{11}$$

Now, from (10) and (11) we get

$$H_{\alpha-2} H_\alpha - H_{\alpha-1}^2 \geq \Delta_{e_2}^{\alpha-2} \Delta_e^{\alpha-2} (\Delta_e - \Delta_{e_2})^2,$$

wherefrom we obtain the required result.

□

Corollary 1. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then*

$$HM_1 \geq 2m^2 + \frac{2\left(\Delta_e - \Delta_{e_2}\right)^2}{\Delta_{e_2} \Delta_e}, \tag{12}$$

with equality if and only if $L(G)$ is regular.

Proof Inequality (12) is a direct consequence of (7) for $\alpha = 1$ and the following equalities

$$M_1 = \sum_{i \sim j} (d_i + d_j) = \sum_{i=1}^{m} (d(e_i)+2)$$

and

$$H = 2H_{-1} = \sum_{i \sim j} \frac{2}{d_i + d_j} = \sum_{i=1}^{m} \frac{2}{d(e_i)+2}.$$

Remark 1. *Since*

$$HM_1 \geq 2m^2 + \frac{2\left(\Delta_e - \Delta_{e_2}\right)^2}{\Delta_{e_2} \Delta_e} \geq 2m^2,$$

the inequality (12) *is stronger than* (6).

Corollary 2. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then*

$$F \geq \frac{M_1^2}{m} - 2M_2 + \frac{\left(\Delta_e - \Delta_{e_2}\right)^2}{m}, \tag{13}$$

with equality if and only if $L(G)$ is regular.

Proof The above inequality is obtained from (7) for $\alpha = 2$ and equality

$$H_2 = F + 2M_2 = \sum_{i\sim j}(d_i + d_j)^2 = \sum_{i=1}^{m}(d(e_i) + 2)^2.$$

Remark 2. *Since*

$$F \geq \frac{M_1^2}{m} - 2M_2 + \frac{(\Delta_e - \Delta_{e_2})^2}{m} \geq \frac{M_1^2}{m} - 2M_2,$$

the inequality (13) *is stronger than* (5).

Corollary 3. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then*

$$F \geq \frac{M_1^2}{2m} + \frac{(\Delta_e - \Delta_{e_2})^2}{2m}. \tag{14}$$

Equality holds if and only if G is regular.

Proof The inequality (14) is obtained from (13) and inequality $F \geq 2M_2$.

Remark 3. *Since*

$$F \geq \frac{M_1^2}{2m} + \frac{(\Delta_e - \Delta_{e_2})^2}{2m} \geq \frac{M_1^2}{2m},$$

the inequality (14) *is stronger than* (4).

Theorem 2. *Let G be a simple connected graph with n vertices and $m \geq 3$ edges. Then*

$$\left(H_{\alpha-2} - \delta_e^{\alpha-2}\right)\left(H_\alpha - \delta_e^{\alpha}\right) \geq \left(H_{\alpha-1} - \delta_e^{\alpha-1}\right)^2 + \Delta_{e_2}^{\alpha-2}\Delta_e^{\alpha-2}\left(\Delta_e - \Delta_{e_2}\right)^2.$$

Equality holds if and only if $L(G)$ is regular graph.

Proof According to (9) we have that

$$\sum_{i=1}^{m-1} p_i \sum_{i=1}^{m-1} p_i a_i b_i - \sum_{i=1}^{m-1} p_i a_i \sum_{i=1}^{m-1} p_i b_i \geq p_1 p_2 (a_1 - a_2)(b_1 - b_2).$$

Putting $p_i = (d(e_i) + 2)^{\alpha-2}$, $a_i = b_i = d(e_i) + 2$, $i = 1, 2, \ldots, m-1$, in this inequality, we get what is stated.

Corollary 4. *Let G be a simple connected graph with n vertices and $m \geq 3$ edges. Then*

$$F \geq \delta_e^2 + \frac{(M_1 - \delta_e)^2}{m-1} - 2M_2 + \frac{(\Delta_e - \Delta_{e_2})^2}{m-1},$$

with equality if and only if $L(G)$ is regular.

Corollary 5. *Let G be a simple connected graph with n vertices and $m \geq 3$ edges. Then*

$$F \geq \frac{1}{2}\delta_e^2 + \frac{(M_1 - \delta_e)^2}{2(m-1)} + \frac{(\Delta_e - \Delta_{e_2})^2}{2(m-1)},$$

with equality if and only if G is regular.

Corollary 6. *Let G be a simple connected graph with n vertices and $m \geq 3$ edges. Then*

$$\left(\frac{1}{2}H - \frac{1}{\delta_e}\right)(M_1 - \delta_e) \geq (m-1)^2 + \frac{(\Delta_e - \Delta_{e_2})^2}{\Delta_e \Delta_{e_2}},$$

with equality if and only if $L(G)$ is regular.

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