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Computing the characteristic polynomials of rooted trees and the energies of Bethe trees

Ivan Damnjanović[a]

[a]*University of Niš, Faculty of Electronic Engineering & Diffine LLC*

Abstract. The mathematical problem of computing the characteristic polynomial of a tree is quite an old one, hence there are many known recursive formulae that deal with this matter. One of these methods is based on assigning a rational function to each vertex in a bottom-up manner, as described by Jacobs and Trevisan [Congr. Numer. 134 (1998) 139–145]. We improve the existing results by giving a shorter and more concise proof of an extension of this method. Furthermore, we implement the aforementioned formula in order to investigate the spectral properties of balanced trees, with a special focus on the Bethe trees, as already done so by Heydari and Taeri [Linear Algebra Appl. 429 (2008) 1744–1757], as well as Robbiano and Trevisan [Comput. Math. with Appl. 59 (2010) 3039–3044]. We improve those results by demonstrating a quicker way to prove the formula that computes the characteristic polynomial of a balanced tree. Finally, we use a technique inspired by Damnjanović et al. [Linear Algebra Appl., doi: 10.1016/j.laa.2022.10.020] in order to obtain a fully computed expression for the energy of any Bethe tree, thereby extending the existing results.

1. Introduction

Let G be a simple graph. We will use $V(G)$ and $n(G)$ to represent the vertex set and order of this graph, respectively. Moreover, we will denote the degree of each vertex $v \in V(G)$ by $d(v)$. Also, we will signify the adjacency and Laplacian matrix of the graph G by $A(G)$ and $L(G)$, respectively. Here, we assume that the rows and columns of $A(G)$ and $L(G)$ correspond to the vertices $u_1, u_2, \ldots, u_{n(G)} \in V(G)$, in this order. We shall denote the characteristic polynomials of these two matrices by $P(G, x) = \det(xI - A(G))$ and $Q(G, x) = \det(xI - L(G))$. Also, we will use $\sigma^*(G)$ to signify the set of all the distinct eigenvalues of $A(G)$, as well as $\mathcal{E}(G)$ in order to denote the energy of the graph G. Here, the graph energy represents the sum of absolute values of all the eigenvalues of $A(G)$, as introduced by Gutman in [5].

We know that a tree represents a simple graph which is both acyclic and connected. Let T be a rooted tree whose root is denoted by $r(T)$. We will enumerate the levels of T by $1, 2, 3, \ldots, l(T)$ so that $r(T)$ is located on level 1 and $l(T) - 1$ represents the eccentricity of $r(T)$. Also, we will use $V(T, j)$ and $n(T, j)$ to signify the set of all of the vertices located on level j and their total number, respectively, for all the $1 \le j \le l(T)$. For

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Email address: ivan.damnjanovic@elfak.ni.ac.rs (Ivan Damnjanović)

convenience, we will also define $n(T, 0) = 0$. Finally, we shall use $c(v)$ to denote the set of all of the children of some given vertex $v \in V(T)$.

The computation of the characteristic polynomial $P(T, x)$ of a tree T is a very old mathematical problem, hence there exist many different recurrent formulae regarding this matter proposed by various authors. For example, Mohar [6, Lemma 2] has disclosed a recurrent expression based on polynomials, as have Tinhofer and Schreck [7, Theorem 1]. Here, we focus on the concise result obtained by Jacobs and Trevisan [1], which displays how the characteristic polynomial of a tree can be determined by assigning a rational function to each vertex in a bottom-up manner. We elaborate the said method in the following corollary.

Corollary 1. *Let T be an arbitrary rooted tree. If we recursively assign a rational function $\mathcal{G}(v, x) \in \mathbb{Z}(x)$ to each of its vertices $v \in V(T)$ by using the expression*

$$\mathcal{G}(v, x) = x - \sum_{w \in c(v)} \frac{1}{\mathcal{G}(w, x)} \qquad (\forall v \in V(T)), \tag{1}$$

then

$$P(T, x) = \prod_{v \in V(T)} \mathcal{G}(v, x). \tag{2}$$

In this paper, we provide a quicker and more concise proof of the given method. In fact, we shall prove a more general theorem that enables us to determine the characteristic polynomial of various other tree-based matrices, such as the Laplacian matrix. The modification of Corollary 1 that works with the said matrix is displayed in the next corollary.

Corollary 2. *Let T be an arbitrary rooted tree. If we recursively assign a rational function $\mathcal{H}(v, x) \in \mathbb{Z}(x)$ to each of its vertices $v \in V(T)$ by using the expression*

$$\mathcal{H}(v, x) = x - d(v) - \sum_{w \in c(v)} \frac{1}{\mathcal{H}(w, x)} \qquad (\forall v \in V(T)), \tag{3}$$

then

$$Q(T, x) = \prod_{v \in V(T)} \mathcal{H}(v, x). \tag{4}$$

As we can see, Corollaries 1 and 2 provide a quick way to compute the characteristic polynomials of the adjacency and Laplacian matrix of a rooted tree, as well as the corresponding spectra. In fact, these methods become especially convenient when we are dealing with a rooted tree that has a regular structure. For example, a balanced tree is considered to be a rooted tree such that all of the vertices on the same level have an equal degree. In this paper, we will give a quick demonstration of how the aforementioned methods can be used in order to investigate the spectral properties of balanced trees, thereby improving some other results based on different computational techniques, such as the ones from [2, 8–10].

Finally, we turn our focus on the Bethe trees. We shall consider the Bethe tree $\mathcal{B}_{d,k}$ to be the balanced tree such that

- $l(\mathcal{B}_{d,k}) = k$;
- each vertex has $d - 1$ children, besides the vertices on the last level, which obviously have none;

as defined by Heilmann and Lieb in [11]. Here, we investigate the spectral properties of these trees, as have some other authors before us, such as Robbiano and Trevisan [3]. Moreover, we extend these results by implementing a technique inspired by Damnjanović et al. [4] in order to provide a fully computed expression for the energy of any Bethe tree.

The remainder of this paper is structured as follows. In Section 2, we state and provide a detailed proof of a theorem which explains how the characteristic polynomials of a wide array of matrices can be computed by implementing the aforementioned method based on assigning rational functions. Subsequently, Section 3 will show how the results obtained in Section 2 can be used with the purpose of investigating the spectral properties of balanced trees. Here, we shall derive an expression that determines the characteristic polynomial and set of distinct eigenvalues of each balanced tree. Finally, Section 4 will rely on the previously derived results in order to investigate the spectral properties of Bethe trees. Within this section, we shall give a fully computed expression for the energies of Bethe trees.

2. Assigned rational functions

Let $\beta(u_1), \beta(u_2), \dots, \beta(u_{n(T)})$ be an arbitrarily chosen sequence of integers which correspond to the vertices $u_1, u_2, \dots, u_{n(T)}$ of a rooted tree T, respectively. For such a given sequence, we will define two additional matrices

$$B_1(T) = A(T) + \operatorname{diag}(\beta(u_1), \beta(u_2), \dots, \beta(u_{n(T)})), \tag{5}$$

$$B_2(T) = -A(T) + \operatorname{diag}(\beta(u_1), \beta(u_2), \dots, \beta(u_{n(T)})). \tag{6}$$

In this paper, we introduce a recursive formula which helps compute the characteristic polynomials of the $B_1(T)$ and $B_2(T)$ matrices for any given β-sequence. This formula is based on assigning a rational function to each vertex of a rooted tree in the bottom-up manner, by using the rational functions previously assigned to its children. The corresponding theorem is given below.

Theorem 1. *Let T be any rooted tree with an arbitrarily chosen β-sequence. If we recursively assign a rational function $\mathcal{F}(v, x) \in \mathbb{Z}(x)$ to each of its vertices $v \in V(T)$ by using the expression*

$$\mathcal{F}(v, x) = x - \beta(v) - \sum_{w \in c(v)} \frac{1}{\mathcal{F}(w, x)} \qquad (\forall v \in V(T)), \tag{7}$$

then

$$\det(xI - B_1(T)) = \det(xI - B_2(T)) = \prod_{v \in V(T)} \mathcal{F}(v, x). \tag{8}$$

First of all, it is clear how Theorem 1 can be implemented in order to yield both Corollary 1 and 2.

Proof of Corollary 1. If we set $\beta(v) = 0$ for all the $v \in V(T)$, we then get $B_1(T) = A(T)$, according to Eq. (5). This means that $P(T, x) = \det(xI - B_1(T))$. By comparing Eq. (1) to Eq. (7), we conclude that Eq. (2) immediately follows from Eq. (8). □

Proof of Corollary 2. By setting $\beta(v) = d(v)$ for all the $v \in V(T)$, we obtain $B_2(T) = L(T)$, according to Eq. (6). This implies $Q(T, x) = \det(xI - B_2(T))$. By comparing Eq. (3) to Eq. (7), we see that Eq. (4) follows directly from Eq. (8). □

In the remainder of this section, we will give a complete proof of Theorem 1. In order to make the logical reasoning more concise and easier to follow, we will start off with some preliminary remarks, then state and prove two auxiliary lemmas which will help us finish the entire proof afterwards.

First of all, it is clear that the matrix $B(T)$ is real and symmetric. Given the fact that any permutation matrix $P \in \mathbb{R}^{n(T) \times n(T)}$ is orthogonal, the matrix $P^T B(T) P$ must also be real and symmetric, as well as similar to $B(T)$. This means that regardless of how we order the vertices of T, the corresponding matrix $B(T)$ will have the same characteristic polynomial. Without loss of generality, we will assume that the vertices $u_1, u_2, \dots, u_{n(T)}$ which correspond to the rows and columns of $B(T)$, in this order, are such that the first

$n(T, l(T))$ of them are all from level $l(T)$, then the following $n(T, l(T)-1)$ are all from level $l(T)-1$, and so on. Bearing this in mind, the matrix $A(T)$ obtains the tridiagonal block form

$$A(T) = \begin{bmatrix} O & D_{l(T)} & O & \cdots & O & O \\ D_{l(T)}^T & O & D_{l(T)-1} & \cdots & O & O \\ O & D_{l(T)-1}^T & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & O & D_2 \\ O & O & O & \cdots & D_2^T & O \end{bmatrix},$$

where the $D_j \in \mathbb{Z}^{n(T,j)\times n(T,j-1)}$, $j = \overline{2, l(T)}$ are all binary and have exactly one 1 per row. This value of 1 signifies which vertex from level $j-1$ is the unique parent of each vertex from level j.

Consequently, we get

$$B_1(T) = \begin{bmatrix} C_{l(T)} & D_{l(T)} & O & \cdots & O & O \\ D_{l(T)}^T & C_{l(T)-1} & D_{l(T)-1} & \cdots & O & O \\ O & D_{l(T)-1}^T & C_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & C_2 & D_2 \\ O & O & O & \cdots & D_2^T & C_1 \end{bmatrix},$$

$$B_2(T) = \begin{bmatrix} C_{l(T)} & -D_{l(T)} & O & \cdots & O & O \\ -D_{l(T)}^T & C_{l(T)-1} & -D_{l(T)-1} & \cdots & O & O \\ O & -D_{l(T)-1}^T & C_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & C_2 & -D_2 \\ O & O & O & \cdots & -D_2^T & C_1 \end{bmatrix},$$

where the $C_j \in \mathbb{Z}^{n(T,j)\times n(T,j)}$, $j = \overline{1, l(T)}$ matrices are all diagonal and such that the element $\beta(v)$ corresponds to the vertex $v \in V(T)$ in the appropriate matrix. This immediately leads us to

$$\det(xI - B_1(T)) = \begin{vmatrix} xI - C_{l(T)} & -D_{l(T)} & O & \cdots & O & O \\ -D_{l(T)}^T & xI - C_{l(T)-1} & -D_{l(T)-1} & \cdots & O & O \\ O & -D_{l(T)-1}^T & xI - C_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & xI - C_2 & -D_2 \\ O & O & O & \cdots & -D_2^T & xI - C_1 \end{vmatrix}, \tag{9}$$

as well as

$$\det(xI - B_2(T)) = \begin{vmatrix} xI - C_{l(T)} & D_{l(T)} & O & \cdots & O & O \\ D_{l(T)}^T & xI - C_{l(T)-1} & D_{l(T)-1} & \cdots & O & O \\ O & D_{l(T)-1}^T & xI - C_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & xI - C_2 & D_2 \\ O & O & O & \cdots & D_2^T & xI - C_1 \end{vmatrix}. \tag{10}$$

Here, the matrices $xI - B_1(T)$ and $xI - B_2(T)$ are such that all of their elements are integer polynomials in x, i.e. members of the integral domain $\mathbb{Z}[x]$. From abstract algebra we know that the corresponding field of fractions of $\mathbb{Z}[x]$ is actually the field of rational functions with integer coefficients, i.e. $\mathbb{Z}(x)$. Henceforth we shall interpret all of the elements of $xI - B_1(T)$ and $xI - B_2(T)$ as members of $\mathbb{Z}(x)$, and thus view $xI - B_1(T)$ and $xI - B_2(T)$ as matrices over the field $\mathbb{Z}(x)$.

Although none of the positive degree polynomials are invertible in $\mathbb{Z}[x]$, they are all invertible in $\mathbb{Z}(x)$. This is the primary reason why the given approach is useful. It will enable us to perform certain block row matrix transformations on $xI - B_1(T)$ and $xI - B_2(T)$ in order to compute the necessary characteristic polynomials, as we shall soon see. We now state and prove two auxiliary lemmas which are necessary to complete the proof of Theorem 1.

Lemma 1. *For each vertex $v \in V(T)$, the assigned rational function $\mathcal{F}(v, x)$ can be represented as a fraction of polynomials*

$$\mathcal{F}(v, x) = \frac{\mathcal{F}_1(v, x)}{\mathcal{F}_2(v, x)},$$

so that $\mathcal{F}_1(v, x), \mathcal{F}_2(v, x) \in \mathbb{Z}[x]$ are both monic polynomials which satisfy

$$\deg \mathcal{F}_1(v, x) = \deg \mathcal{F}_2(v, x) + 1.$$

Proof. We will prove the lemma via mathematical induction. If we pick an arbitrary vertex $v \in n(T, l(T))$, we know that it must be a leaf, hence its assigned rational function is the linear polynomial $x - \beta(v)$, according to Eq. (7). This rational function obviously satisfies the lemma statement, given the fact that $\mathcal{F}(v, x) = \frac{x - \beta(v)}{1}$. Now suppose that the statement holds for all of the vertices on level $j + 1$, where $1 \le j \le l(T) - 1$. We will complete the proof by showing that it must hold for each vertex on level j as well.

Let $v \in V(T, j)$ be an arbitrary vertex on level j. From the induction hypothesis, we know that all of its children satisfy the lemma statement, i.e. for each $w \in c(v)$ we have

$$\mathcal{F}(w, x) = \frac{\mathcal{F}_1(w, x)}{\mathcal{F}_2(w, x)},$$

where $\mathcal{F}_1(w, x), \mathcal{F}_2(w, x) \in \mathbb{Z}[x]$ are monic polynomials such that

$$\deg \mathcal{F}_1(w, x) = \deg \mathcal{F}_2(w, x) + 1.$$

We further have

$$\mathcal{F}(v, x) = x - \beta(v) - \sum_{w \in c(v)} \frac{1}{\mathcal{F}(w, x)} = x - \beta(v) - \sum_{w \in c(v)} \frac{\mathcal{F}_2(w, x)}{\mathcal{F}_1(w, x)}$$
$$= \frac{(x - \beta(v)) \prod_{w \in c(v)} \mathcal{F}_1(w, x) - \sum_{w \in c(v)} \left(\mathcal{F}_2(w, x) \prod_{t \in c(v), t \neq w} \mathcal{F}_1(t, x) \right)}{\prod_{w \in c(v)} \mathcal{F}_1(w, x)}.$$

If we write

$$\mathcal{F}_1(v, x) = (x - \beta(v)) \prod_{w \in c(v)} \mathcal{F}_1(w, x) - \sum_{w \in c(v)} \left(\mathcal{F}_2(w, x) \prod_{t \in c(v), t \neq w} \mathcal{F}_1(t, x) \right),$$
$$\mathcal{F}_2(v, x) = \prod_{w \in c(v)} \mathcal{F}_1(w, x),$$

we then obtain $\mathcal{F}(v, x) = \frac{\mathcal{F}_1(v, x)}{\mathcal{F}_2(v, x)}$, where $\mathcal{F}_1(v, x), \mathcal{F}_2(v, x) \in \mathbb{Z}[x]$ are clearly both monic and satisfy $\deg \mathcal{F}_1(v, x) = \deg \mathcal{F}_2(v, x) + 1$ as well. This implies that the lemma statement holds for an arbitrary $v \in V(T, j)$, which completes the proof via mathematical induction. □

Lemma 1 directly shows that $\mathcal{F}(v, x) \in \mathbb{Z}(x)$ is invertible for each $v \in V(T)$. This is important because it makes the definition itself of the assigned rational functions valid, given the fact that the rational function assigned to each vertex demands that the rational functions assigned to all of its children have an inverse, according to Eq. (7). We will make further use of this property in the next lemma.

Lemma 2. *For each $1 \le j \le l(T)$, let R_j be the diagonal square matrix of order $n(T, j)$ over the field $\mathbb{Z}(x)$, such that the diagonal entry corresponding to the vertex $v \in V(T, j)$ is equal to $\mathcal{F}(v, x)$, according to the predetermined order of vertices on level j. Then, the following equation holds*

$$R_j = xI - C_j - D_{j+1}^T R_{j+1}^{-1} D_{j+1}, \tag{11}$$

for all the $j = \overline{1, l(T) - 1}$.

Proof. First of all, the matrix R_{j+1} is invertible, due to the fact that it is diagonal, and all of its diagonal entries are invertible, as a direct consequence of Lemma 1. This means that the formula Eq. (11) is valid to begin with. Furthermore, R_{j+1}^{-1} must be equal to the diagonal matrix such that the element corresponding to the vertex $v \in V(T, j+1)$ is $\dfrac{1}{\mathcal{F}(v, x)}$. If we denote $Z_j = D_{j+1}^T R_{j+1}^{-1} D_{j+1}$, from basic matrix multiplication we get

$$[Z_j]_{\alpha,\beta} = \sum_{i=1}^{n(T,j+1)} \sum_{h=1}^{n(T,j+1)} [D_{j+1}^T]_{\alpha,i} \, [R_{j+1}^{-1}]_{i,h} \, [D_{j+1}]_{h,\beta} = \sum_{i,h=1}^{n(T,j+1)} [D_{j+1}^T]_{\alpha,i} \, [R_{j+1}^{-1}]_{i,h} \, [D_{j+1}]_{h,\beta} \, . \tag{12}$$

Given the fact that the matrix R_{j+1}^{-1} is diagonal, we conclude that all of the sum terms in Eq. (12) where $i \neq h$ amount to zero. This leads us to

$$[Z_j]_{\alpha,\beta} = \sum_{i=1}^{n(T,j+1)} [D_{j+1}^T]_{\alpha,i} \, [R_{j+1}^{-1}]_{i,i} \, [D_{j+1}]_{i,\beta} = \sum_{i=1}^{n(T,j+1)} [D_{j+1}]_{i,\alpha} \, [D_{j+1}]_{i,\beta} \, [R_{j+1}^{-1}]_{i,i} \, . \tag{13}$$

Since the matrix D_{j+1} has exactly one non-zero element per row, it immediately follows from Eq. (13) that $[Z_j]_{\alpha,\beta} = 0$ whenever $\alpha \neq \beta$. This means that the matrix Z_j is necessarily diagonal. We know that D_{j+1} is also binary, which means that

$$[Z_j]_{\alpha,\alpha} = \sum_{i=1}^{n(T,j+1)} [D_{j+1}]_{i,\alpha} \, [D_{j+1}]_{i,\alpha} \, [R_{j+1}^{-1}]_{i,i} = \sum_{i=1}^{n(T,j+1)} [D_{j+1}]_{i,\alpha} \, [R_{j+1}^{-1}]_{i,i} \, . \tag{14}$$

From Eq. (14) it becomes easy to see that Z_j must be the diagonal matrix of order $n(T, j)$ such that the diagonal entry corresponding to the vertex $v \in V(T, j)$ is equal to $\displaystyle\sum_{w \in c(v)} \frac{1}{\mathcal{F}(w, x)}$. In that case, the matrix $xI - C_j - D_{j+1}^T R_{j+1}^{-1} D_{j+1}$ must be diagonal and such that the diagonal entry corresponding to the vertex $v \in V(T, j)$ is equal to $x - \beta(v) - \displaystyle\sum_{w \in c(v)} \frac{1}{\mathcal{F}(w, x)}$. Hence, this matrix must be equal to R_j. □

We now have all of the tools necessary to complete the proof of Theorem 1.

Proof of Theorem 1. We will prove only the

$$\det(xI - B_1(T)) = \prod_{v \in V(T)} \mathcal{F}(v, x)$$

part of Eq. (8). The second half is proved absolutely analogously, so the corresponding proof will be omitted.

Given the fact that all of the vertices on level $l(T)$ are leaves, it is clear that $R_{l(T)} = xI - C_{l(T)}$. From Eq. (9) we get

$$\det(xI - B_1(T)) = \begin{vmatrix} R_{l(T)} & -D_{l(T)} & O & \cdots & O & O \\ -D_{l(T)}^T & xI - C_{l(T)-1} & -D_{l(T)-1} & \cdots & O & O \\ O & -D_{l(T)-1}^T & xI - C_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & xI - C_2 & -D_2 \\ O & O & O & \cdots & -D_2^T & xI - C_1 \end{vmatrix}.$$

Due to Lemma 2, we know that the matrix $R_{l(T)}$ is invertible. We can now multiply the first block row with $D_{l(T)}^T R_{l(T)}^{-1}$ to the left and add it to the second block row, in order to obtain

$$\det(xI - B_1(T)) = \begin{vmatrix} R_{l(T)} & -D_{l(T)} & O & \cdots & O & O \\ O & R_{l(T)-1} & -D_{l(T)-1} & \cdots & O & O \\ O & -D_{l(T)-1}^T & xI - C_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & xI - C_2 & -D_2 \\ O & O & O & \cdots & -D_2^T & xI - C_1 \end{vmatrix}.$$

Here, we have used the fact that $R_{l(T)-1} = xI - C_{l(T)-1} - D_{l(T)}^T R_{l(T)}^{-1} D_{l(T)}$, which follows from Lemma 2. We can now multiply the second block row with $D_{l(T)-1}^T R_{l(T)-1}^{-1}$ to the left and add it to the third block row, thus getting

$$\det(xI - B_1(T)) = \begin{vmatrix} R_{l(T)} & -D_{l(T)} & O & \cdots & O & O \\ O & R_{l(T)-1} & -D_{l(T)-1} & \cdots & O & O \\ O & O & R_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & xI - C_2 & -D_2 \\ O & O & O & \cdots & -D_2^T & xI - C_1 \end{vmatrix},$$

thanks to $R_{l(T)-2} = xI - C_{l(T)-2} - D_{l(T)-1}^T R_{l(T)-1}^{-1} D_{l(T)-1}$, which is also a consequence of Lemma 2. By repeating the same process until the last row, we conclude that

$$\det(xI - B_1(T)) = \begin{vmatrix} R_{l(T)} & -D_{l(T)} & O & \cdots & O & O \\ O & R_{l(T)-1} & -D_{l(T)-1} & \cdots & O & O \\ O & O & R_{l(T)-2} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & R_2 & -D_2 \\ O & O & O & \cdots & O & R_1 \end{vmatrix}.$$

It immediately follows that

$$\det(xI - B_1(T)) = \prod_{j=1}^{l(T)} \det(R_j).$$

However, from the definition of R_j we know that $\det(R_j) = \prod_{v \in V(T,j)} \mathcal{F}(v, x)$, which directly gives us

$$\det(xI - B_1(T)) = \prod_{j=1}^{l(T)} \prod_{v \in V(T,j)} \mathcal{F}(v, x) = \prod_{v \in V(T)} \mathcal{F}(v, x).$$

□

Remark 1. *It is also possible to allow the β-sequence to be a sequence of real numbers, instead of strictly integers. In this case, Theorem 1 would continue to hold, with the sole difference being that the assigned rational functions $\mathcal{F}(v,x)$ would be members of $\mathbb{R}(x)$ instead of $\mathbb{Z}(x)$.*

3. Spectral properties of balanced trees

For a given balanced tree T, let $c_1, c_2, \ldots, c_{l(T)}$ be the sequence which defines how many children every vertex from each level has. In other words, c_j should represent the number of children of every vertex from level j, for all the $1 \le j \le l(T)$. Here, it is important to note that $c_j = d(v) - 1$ holds for each $v \in V(T, j)$, provided $j > 1$. Also, it is clear that $c_1 = d(r(T))$. In order to resume our investigation of spectral properties regarding balanced trees, we will need the following short lemma.

Lemma 3. *Let T be any balanced tree with an arbitrarily chosen β-sequence such that any two vertices on the same level have equal corresponding β-elements. If we recursively assign a rational function $\mathcal{F}(v,x) \in \mathbb{Z}(x)$ to each vertex $v \in V(T)$ via Eq. (7), then any two vertices on the same level must have equal assigned rational functions.*

Proof. The lemma is straightforward to prove via mathematical induction. The statement obviously holds for the vertices on the last level $l(T)$, given the fact that Eq. (7) amounts to $\mathcal{F}(v,x) = x - \beta(v)$ for any $v \in V(T, l(T))$. Suppose that the statement holds for the vertices on level $j+1$, where $1 \le j \le l(T) - 1$. We will now prove that it must hold for the vertices on level j as well.

Let v be any vertex from level j. Also, let $v_1 \in V(T, j)$ and $v_2 \in V(T, j+1)$ be two arbitrarily chosen fixed vertices. By implementing the induction hypothesis, as well as the fact that $\beta(v) = \beta(v_1)$, Eq. (7) helps us obtain

$$\mathcal{F}(v,x) = x - \beta(v) - \sum_{w \in c(v)} \frac{1}{\mathcal{F}(w,x)} = x - \beta(v_1) - \frac{c_j}{\mathcal{F}(v_2,x)}.$$

It becomes clear that $\mathcal{F}(v,x)$ is the same for any chosen $v \in V(T, j)$, which completes the proof. □

The key observation is that Lemma 3 can be implemented on the assigned rational functions $\mathcal{G}$ and $\mathcal{H}$ defined in Corollaries 1 and 2, respectively. This implies that for any balanced tree T it is much more convenient to view the aforementioned assigned rational functions as sequences $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{l(T)}$ and $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{l(T)}$, such that $\mathcal{G}_j$ and $\mathcal{H}_j$ are assigned to every vertex $v \in V(T, j)$, for all the $1 \le j \le l(T)$. This swiftly leads to the following two corollaries.

Corollary 3. *Let T be an arbitrary balanced tree. If $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{l(T)} \in \mathbb{Z}(x)$ is a sequence of rational functions which is recursively defined via*

$$\begin{aligned} \mathcal{G}_{l(T)} &= x, \\ \mathcal{G}_j &= x - \frac{c_j}{\mathcal{G}_{j+1}} \qquad (\forall j \in \overline{1, l(T)-1}), \end{aligned} \tag{15}$$

then

$$P(T,x) = \prod_{j=1}^{l(T)} \mathcal{G}_j^{n(T,j)}. \tag{16}$$

Corollary 4. *Let T be any nontrivial balanced tree. If $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{l(T)} \in \mathbb{Z}(x)$ is a sequence of rational functions which is recursively defined via*

$$\begin{aligned} \mathcal{H}_{l(T)} &= x - 1, \\ \mathcal{H}_j &= x - (c_j + 1) - \frac{c_j}{\mathcal{H}_{j+1}} \qquad (\forall j \in \overline{2, l(T)-1}), \\ \mathcal{H}_1 &= x - c_1 - \frac{c_1}{\mathcal{H}_2}, \end{aligned} \tag{17}$$

then

$$Q(T,x) = \prod_{j=1}^{l(T)} \mathcal{H}_j^{n(T,j)}. \tag{18}$$

Remark 2. *Corollary 4 does not hold if T is the trivial balanced tree, so the condition that the tree is nontrivial cannot be omitted. This is due to the fact that the trivial tree is such that all of its leaves have no parent, which is the only case when Eq. (17) does not hold.*

Corollaries 3 and 4 follow immediately from Corollaries 1 and 2 by implementing the elaborated consequences of Lemma 3. In fact, Eqs. (16) and (18) can be transformed into simpler forms which avoid rational functions and use only polynomials. This is demonstrated in the following theorem.

Theorem 2. *For an arbitrary balanced tree T, let* $W_0, W_1, W_2, \dots, W_{l(T)} \in \mathbb{Z}[x]$ *be a sequence of polynomials defined via the recurrence relation*

$$\begin{aligned} W_0 &= 1, \\ W_1 &= x, \\ W_j &= xW_{j-1} - c_{l(T)+1-j}W_{j-2} \qquad (\forall j \in \overline{2, l(T)}). \end{aligned} \tag{19}$$

We then have

$$P(T,x) = \prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)-n(T,l(T)-j)}. \tag{20}$$

Also, if T is nontrivial, then for the sequence $Y_0, Y_1, Y_2, \dots, Y_{l(T)} \in \mathbb{Z}[x]$ *of polynomials determined via*

$$\begin{aligned} Y_0 &= 1, \\ Y_1 &= x - 1, \\ Y_j &= (x - c_{l(T)+1-j} - 1)Y_{j-1} - c_{l(T)+1-j}Y_{j-2} \qquad (\forall j \in \overline{2, l(T)-1}), \\ Y_{l(T)} &= (x - c_1)Y_{l(T)-1} - c_1 Y_{l(T)-2}, \end{aligned}$$

we get

$$Q(T,x) = \prod_{j=1}^{l(T)} Y_j^{n(T,l(T)+1-j)-n(T,l(T)-j)}. \tag{21}$$

Proof. We will prove only Eq. (20) due to the fact that Eq. (21) is proved in an analogous manner by implementing Corollary 4 instead of Corollary 3. First of all, we will show via mathematical induction that

$$\mathcal{G}_j = \frac{W_{l(T)+1-j}}{W_{l(T)-j}} \tag{22}$$

holds for all the $1 \le j \le l(T)$, where $\mathcal{G}_j$ is the sequence of assigned rational functions from Corollary 3. It can immediately be seen that Eq. (22) holds for the rational function assigned to the vertices on the last level, since we know that $\mathcal{G}_{l(T)} = x$, as well as $\dfrac{W_1}{W_0} = \dfrac{x}{1} = x$. Now suppose that Eq. (22) is true for the rational function assigned to the vertices on level $j+1$, where $1 \le j \le l(T) - 1$. We will complete this part of the proof by showing that Eq. (22) must necessarily be valid for the rational function assigned to the vertices on level j as well.

From Eq. (15) we quickly obtain

$$\mathcal{G}_j = x - \frac{c_j}{\mathcal{G}_{j+1}} = x - c_j \frac{W_{l(T)-j-1}}{W_{l(T)-j}} = \frac{xW_{l(T)-j} - c_j W_{l(T)-j-1}}{W_{l(T)-j}} = \frac{W_{l(T)+1-j}}{W_{l(T)-j}},$$

which implies that Eq. (22) holds for the rational function assigned to the vertices on level j, as needed.

Taking into consideration Eq. (22) along with Eq. (16), we conclude that

$$P(T,x) = \prod_{j=1}^{l(T)} \mathcal{G}_j^{n(T,j)} = \prod_{j=1}^{l(T)} \left(\frac{W_{l(T)+1-j}}{W_{l(T)-j}} \right)^{n(T,j)} = \frac{\prod_{j=1}^{l(T)} W_{l(T)+1-j}^{n(T,j)}}{\prod_{j=1}^{l(T)} W_{l(T)-j}^{n(T,j)}}.$$

This means that

$$P(T,x) = \frac{\prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)}}{\prod_{j=0}^{l(T)-1} W_j^{n(T,l(T)-j)}}.$$

By using $W_0 = 1$, we further get

$$P(T,x) = \frac{\prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)}}{W_0^{n(T,l(T))} \prod_{j=1}^{l(T)-1} W_j^{n(T,l(T)-j)}} = \frac{\prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)}}{\prod_{j=1}^{l(T)-1} W_j^{n(T,l(T)-j)}}.$$

Finally, since $n(T,0) = 0$, we reach

$$P(T,x) = \frac{\prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)}}{W_{l(T)}^{n(T,0)} \prod_{j=1}^{l(T)-1} W_j^{n(T,l(T)-j)}} = \frac{\prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)}}{\prod_{j=1}^{l(T)} W_j^{n(T,l(T)-j)}} = \prod_{j=1}^{l(T)} W_j^{n(T,l(T)+1-j)-n(T,l(T)-j)}.$$

□

Theorem 2 has some interesting direct consequences. For example, Eq. (20) makes it easy to determine $\sigma^*(T)$, as shown in the next corollary.

Corollary 5. *Let T be an arbitrary balanced tree and let $\Phi \subseteq \mathbb{N}$ be the set*

$$\Phi = \{l(T)\} \cup \{j \in \mathbb{N} \colon 1 \le j \le l(T) - 1, c_{l(T)-j} > 1\}.$$

Then

$$\sigma^*(T) = \bigcup_{j \in \Phi} \{x \in \mathbb{R} \colon W_j(x) = 0\}. \tag{23}$$

Proof. From Eq. (20) we have that $P(T, x)$ can be represented as a product of certain elements of the W_j sequence. Moreover, some W_j appears as a factor of $P(T, x)$ if and only if

$$\begin{aligned} & n(T, l(T) + 1 - j) - n(T, l(T) - j) > 0 \\ \iff \quad & n(T, l(T) + 1 - j) > n(T, l(T) - j). \end{aligned} \tag{24}$$

For $j = l(T)$, this condition is always satisfied, due to the fact that $n(T, 1) = 1$ and $n(T, 0) = 0$. If $1 \le j \le l(T)-1$, we then have

$$n(T, l(T) + 1 - j) = c_{l(T)-j}\, n(T, l(T) - j),$$

which means that Eq. (24) is equivalent to $c_{l(T)-j} > 1$. Thus, we get that W_j appears as a factor of $P(T, x)$ if and only if $j \in \Phi$. This clearly means that some real number belongs to $\sigma^*(T) \subseteq \mathbb{R}$ if and only if it is a root of at least one of the polynomials W_j where $j \in \Phi$. The noted observation promptly leads to Eq. (23). □

Remark 3. *It is worth noting that Eq. (21) could be used in order to make a similar conclusion regarding the set of distinct eigenvalues of $L(T)$ and the Y_j sequence of polynomials.*

4. Energy of Bethe trees

In this section we will implement Theorem 2 and Corollary 5 on the Bethe tree $\mathcal{B}_{d,k}$ in order to compute its characteristic polynomial $P(\mathcal{B}_{d,k}, x)$, set of distinct eigenvalues $\sigma^*(\mathcal{B}_{d,k})$ and energy $\mathcal{E}(\mathcal{B}_{d,k})$, for all the $d \ge 2$ and $k \ge 1$. Let $E_0(x, a), E_1(x, a), E_2(x, a), \ldots$ be the sequence of polynomials defined via the recurrence relation

$$\begin{aligned} & E_0(x, a) = 1, \\ & E_1(x, a) = x, \\ & E_j(x, a) = xE_{j-1}(x, a) - aE_{j-2}(x, a) \qquad (\forall j \ge 2), \end{aligned} \tag{25}$$

where $a \in \mathbb{R}$ is a fixed constant. We shall call these polynomials the Dickson polynomials of the second kind, as done so in [12, pp. 9–10]. There are many known properties regarding the polynomials from this sequence, which is very convenient for us, given the fact that

$$W_j = E_j(x, d-1) \qquad (\forall j = \overline{0, l(T)}), \tag{26}$$

where W_j is the corresponding sequence from Theorem 2 when it is applied on the balanced tree $T = \mathcal{B}_{d,k}$. This is not difficult to see, since $T = \mathcal{B}_{d,k}$ implies $c_j = d-1$ for all the $1 \le j \le l(T)-1$, which immediately makes the recurrence relation Eq. (19) defining $W_0, W_1, \ldots, W_{l(T)}$ equivalent to the recurrence relation Eq. (25). This observation leads to the following theorem.

Theorem 3. *For any Bethe tree $\mathcal{B}_{d,k}$, where $d \ge 2$ and $k \ge 1$, we have*

$$P(\mathcal{B}_{d,k}, x) = E_k(x, d-1) \prod_{j=1}^{k-1} E_j(x, d-1)^{(d-2)(d-1)^{k-1-j}}. \tag{27}$$

Proof. If we compare Eq. (27) to Eq. (20) and use Eq. (26), as well as $k = l(T)$, it becomes sufficient to show that $n(T, 1) - n(T, 0) = 1$ and

$$n(T, k + 1 - j) - n(T, k - j) = (d-2)(d-1)^{k-1-j},$$

for all of the $1 \le j \le k-1$. The first fact is obvious, while the second follows directly from $n(T, k + 1 - j) = (d-1)^{k-j}$ and $n(T, k - j) = (d-1)^{k-1-j}$. □

We can now use Corollary 5 together with some known properties about the Dickson polynomials of the second kind in order to determine $\sigma^*(\mathcal{B}_{d,k})$. The full procedure is given in the next corollary.

Corollary 6. *For any Bethe tree $\mathcal{B}_{d,k}$, where $d \geq 3$ and $k \geq 1$, we have*

$$\sigma^*(\mathcal{B}_{d,k}) = \left\{ 2\sqrt{d-1}\cos\left(\frac{h}{j+1}\pi\right) : 1 \leq h \leq j \leq k \right\}. \quad (28)$$

Also, for any $k \geq 1$, we have

$$\sigma^*(\mathcal{B}_{2,k}) = \left\{ 2\cos\left(\frac{h}{k+1}\pi\right) : 1 \leq h \leq k \right\}. \quad (29)$$

Proof. The Bethe tree $\mathcal{B}_{2,k}$ represents a path graph composed of k vertices. It is known (see, for example, [13, pp. 18]) that the spectrum of this graph is composed of the simple eigenvalues

$$2\cos\left(\frac{1}{k+1}\pi\right), 2\cos\left(\frac{2}{k+1}\pi\right), \ldots, 2\cos\left(\frac{k}{k+1}\pi\right),$$

which proves Eq. (29).

On the other hand, it is also known that $E_j(x,a)$ has j distinct simple roots

$$2\sqrt{a}\cos\left(\frac{1}{j+1}\pi\right), 2\sqrt{a}\cos\left(\frac{2}{j+1}\pi\right), \ldots, 2\sqrt{a}\cos\left(\frac{j}{j+1}\pi\right)$$

(see, for example, [12, pp. 9–10]). By implementing Corollary 5, we get that $\Phi = \{1, 2, \ldots, k\}$, since $c_j > 1$ for all of the $1 \leq j \leq k-1$, provided $d \geq 3$. This means that for $d \geq 3$, we have that $\sigma^*(\mathcal{B}_{d,k})$ is composed of the real numbers which are a root of at least one polynomial from the sequence $E_1(x, d-1), E_2(x, d-1), \ldots, E_k(x, d-1)$. However, the set of such real numbers is obviously equal to the set represented in Eq. (28). □

To finish our investigation of spectral properties regarding the Bethe trees, we shall determine the energy of $\mathcal{B}_{d,k}$. Our key result regarding this matter is presented in the following theorem.

Theorem 4. *For an arbitrary Bethe tree $\mathcal{B}_{d,k}$, where $d \geq 3$ and $k \geq 1$, we have*

$$\mathcal{E}(\mathcal{B}_{d,k}) = \sum_{j=1}^{k-1} f_j (d-1)^{k-\frac{1}{2}-j}, \quad (30)$$

where

$$f_j = \begin{cases} 2\csc\left(\frac{\pi}{2j+4}\right) - 2\cot\left(\frac{\pi}{2j+2}\right), & 2 \nmid j, \\ 2\cot\left(\frac{\pi}{2j+4}\right) - 2\csc\left(\frac{\pi}{2j+2}\right), & 2 \mid j. \end{cases}$$

Also, for any $k \geq 1$, we have

$$\mathcal{E}(\mathcal{B}_{2,k}) = \begin{cases} 2\left(\cot\left(\frac{\pi}{2k+2}\right) - 1\right), & 2 \nmid k, \\ 2\left(\csc\left(\frac{\pi}{2k+2}\right) - 1\right), & 2 \mid k. \end{cases} \quad (31)$$

In order to provide a proof of Theorem 4, we will rely on certain properties regarding the $E_j(x,a)$ polynomials. Let $\Psi(E_j(x,a))$ denote the sum of absolute values of all of the roots of $E_j(x,a)$. The upcoming auxiliary lemma solves the problem of giving an explicit formula for $\Psi(E_j(x,a))$.

Lemma 4. *For any Dickson polynomial of the second kind $E_j(x,a)$, we have*

$$\Psi(E_j(x,a)) = \begin{cases} 2\sqrt{a}\left(\cot\left(\dfrac{\pi}{2j+2}\right)-1\right), & 2 \nmid j, \\ 2\sqrt{a}\left(\csc\left(\dfrac{\pi}{2j+2}\right)-1\right), & 2 \mid j. \end{cases}$$

Proof. As discussed earlier, we know that the roots of the polynomial $E_j(x,a)$ are

$$2\sqrt{a}\cos\left(\frac{1}{j+1}\pi\right), 2\sqrt{a}\cos\left(\frac{2}{j+1}\pi\right), \dots, 2\sqrt{a}\cos\left(\frac{j}{j+1}\pi\right).$$

Hence

$$\Psi(E_j(x,a)) = \sum_{h=1}^{j}\left|2\sqrt{a}\cos\left(\frac{h}{j+1}\pi\right)\right| = 2\sqrt{a}\sum_{h=1}^{j}\left|\cos\left(\frac{h}{j+1}\pi\right)\right|.$$

Since $\cos\left(\frac{h}{j+1}\pi\right) > 0$ for $1 \le h < \frac{j+1}{2}$ and $\cos\left(\frac{h}{j+1}\pi\right) = -\cos\left(\frac{j+1-h}{j+1}\pi\right)$ for all the $\frac{j+1}{2} < h \le j$, we can rewrite the last expression as

$$\Psi(E_j(x,a)) = 4\sqrt{a}\sum_{h=1}^{\lfloor \frac{j}{2} \rfloor}\cos\left(\frac{h}{j+1}\pi\right).$$

Let us denote $\zeta = e^{\frac{i\pi}{j+1}}$. It is convenient to replace $\cos\left(\frac{h}{j+1}\pi\right)$ with $\frac{\zeta^h+\zeta^{-h}}{2}$. This gives us:

$$\Psi(E_j(x,a)) = 4\sqrt{a}\sum_{h=1}^{\lfloor \frac{j}{2} \rfloor}\frac{\zeta^h+\zeta^{-h}}{2} = 2\sqrt{a}\sum_{h=1}^{\lfloor \frac{j}{2} \rfloor}\left(\zeta^h+\zeta^{-h}\right) = 2\sqrt{a}\left(\left(\sum_{h=-\lfloor \frac{j}{2} \rfloor}^{\lfloor \frac{j}{2} \rfloor}\zeta^h\right)-1\right).$$

Since $\zeta \ne 1$, we can use the standard formula for summing a geometric progression in order to get

$$\begin{aligned}\Psi(E_j(x,a)) &= 2\sqrt{a}\left(\frac{\sum_{h=0}^{2\lfloor \frac{j}{2} \rfloor}\zeta^h}{\zeta^{\lfloor \frac{j}{2} \rfloor}}-1\right) = 2\sqrt{a}\left(\frac{\zeta^{2\lfloor \frac{j}{2} \rfloor+1}-1}{\zeta^{\lfloor \frac{j}{2} \rfloor}(\zeta-1)}-1\right) \\ &= 2\sqrt{a}\left(\frac{\zeta^{\lfloor \frac{j}{2} \rfloor+1}-\zeta^{-\lfloor \frac{j}{2} \rfloor}}{\zeta-1}-1\right) = 2\sqrt{a}\left(\frac{\left(\zeta^{\lfloor \frac{j}{2} \rfloor+1}-\zeta^{-\lfloor \frac{j}{2} \rfloor}\right)\left(\frac{1}{\zeta}-1\right)}{(\zeta-1)\left(\frac{1}{\zeta}-1\right)}-1\right).\end{aligned}$$

By taking into consideration that

$$\left(\zeta^{\lfloor \frac{j}{2} \rfloor+1}-\zeta^{-\lfloor \frac{j}{2} \rfloor}\right)\left(\frac{1}{\zeta}-1\right) = \zeta^{\lfloor \frac{j}{2} \rfloor}+\zeta^{-\lfloor \frac{j}{2} \rfloor}-\zeta^{\lfloor \frac{j}{2} \rfloor+1}-\zeta^{-\lfloor \frac{j}{2} \rfloor-1} = 2\cos\left(\frac{\lfloor \frac{j}{2} \rfloor}{j+1}\pi\right)-2\cos\left(\frac{\lfloor \frac{j}{2} \rfloor+1}{j+1}\pi\right)$$

and

$$(\zeta-1)\left(\frac{1}{\zeta}-1\right) = 2-\left(\zeta+\frac{1}{\zeta}\right) = 2-2\cos\left(\frac{1}{j+1}\pi\right),$$

we conclude that

$$\begin{aligned}\Psi(E_j(x,a)) &= 2\sqrt{a}\left(\frac{2\cos\left(\frac{\lfloor\frac{j}{2}\rfloor}{j+1}\pi\right) - 2\cos\left(\frac{\lfloor\frac{j}{2}\rfloor+1}{j+1}\pi\right)}{2 - 2\cos\left(\frac{1}{j+1}\pi\right)} - 1\right)\\ &= 2\sqrt{a}\left(\frac{\cos\left(\frac{\lfloor\frac{j}{2}\rfloor}{j+1}\pi\right) - \cos\left(\frac{\lfloor\frac{j}{2}\rfloor+1}{j+1}\pi\right)}{1 - \cos\left(\frac{1}{j+1}\pi\right)} - 1\right). \end{aligned} \tag{32}$$

If j is odd, then $\lfloor\frac{j}{2}\rfloor = \frac{j-1}{2}$ and $\frac{\lfloor\frac{j}{2}\rfloor+1}{j+1}\pi = \frac{\pi}{2}$, which transforms Eq. (32) into

$$\begin{aligned}\Psi(E_j(x,a)) &= 2\sqrt{a}\left(\frac{\cos\left(\frac{j-1}{j+1}\cdot\frac{\pi}{2}\right)}{1-\cos\left(\frac{1}{j+1}\pi\right)} - 1\right) = 2\sqrt{a}\left(\frac{\sin\left(\frac{2}{j+1}\cdot\frac{\pi}{2}\right)}{2\sin^2\left(\frac{\pi}{2j+2}\right)} - 1\right) = 2\sqrt{a}\left(\frac{\sin\left(\frac{\pi}{j+1}\right)}{2\sin^2\left(\frac{\pi}{2j+2}\right)} - 1\right)\\ &= 2\sqrt{a}\left(\frac{2\sin\left(\frac{\pi}{2j+2}\right)\cos\left(\frac{\pi}{2j+2}\right)}{2\sin^2\left(\frac{\pi}{2j+2}\right)} - 1\right) = 2\sqrt{a}\left(\cot\left(\frac{\pi}{2j+2}\right) - 1\right).\end{aligned}$$

If j is even, then $\lfloor\frac{j}{2}\rfloor = \frac{j}{2}$, as well as $\cos\left(\frac{\lfloor\frac{j}{2}\rfloor+1}{j+1}\pi\right) = -\cos\left(\frac{\lfloor\frac{j}{2}\rfloor}{j+1}\pi\right)$, which gives

$$\Psi(E_j(x,a)) = 2\sqrt{a}\left(\frac{2\cos\left(\frac{j}{j+1}\cdot\frac{\pi}{2}\right)}{1-\cos\left(\frac{1}{j+1}\pi\right)} - 1\right) = 2\sqrt{a}\left(\frac{2\sin\left(\frac{1}{j+1}\cdot\frac{\pi}{2}\right)}{2\sin^2\left(\frac{\pi}{2j+2}\right)} - 1\right) = 2\sqrt{a}\left(\csc\left(\frac{\pi}{2j+2}\right) - 1\right).$$

□

We are now able to implement Lemma 4 in order to finish the computation of $\mathcal{E}(\mathcal{B}_{d,k})$.

Proof of Theorem 4. First of all, from Eq. (27) it is clear that $P(\mathcal{B}_{2,k}, x) = E_k(x, 1)$ for every $k \geq 1$. This means that $\mathcal{E}(\mathcal{B}_{2,k}) = \Psi(E_k(x, 1))$. Hence, Eq. (31) follows immediately from Lemma 4.

Now, we will suppose that $d \geq 3$ and prove Eq. (30). From Eq. (27), we obtain

$$\mathcal{E}(\mathcal{B}_{d,k}) = \Psi(E_k(x, d-1)) + \sum_{j=1}^{k-1}(d-2)(d-1)^{k-1-j}\Psi(E_j(x, d-1)).$$

However,

$$\begin{aligned}&\sum_{j=1}^{k-1}(d-2)(d-1)^{k-1-j}\Psi(E_j(x,d-1)) =\\ &\quad= \sum_{j=1}^{k-1}((d-1)^{k-j} - (d-1)^{k-1-j})\Psi(E_j(x,d-1))\\ &\quad= \sum_{j=1}^{k-1}(d-1)^{k-j}\Psi(E_j(x,d-1)) - \sum_{j=1}^{k-1}(d-1)^{k-1-j}\Psi(E_j(x,d-1))\\ &\quad= \sum_{j=0}^{k-2}(d-1)^{k-1-j}\Psi(E_{j+1}(x,d-1)) - \sum_{j=1}^{k-1}(d-1)^{k-1-j}\Psi(E_j(x,d-1)).\end{aligned}$$

Given the fact that

$$\Psi(E_k(x, d-1)) + \sum_{j=0}^{k-2} (d-1)^{k-1-j} \Psi(E_{j+1}(x, d-1)) = \sum_{j=0}^{k-1} (d-1)^{k-1-j} \Psi(E_{j+1}(x, d-1)),$$

we further obtain

$$\mathcal{E}(\mathcal{B}_{d,k}) = \sum_{j=0}^{k-1} (d-1)^{k-1-j} \Psi(E_{j+1}(x, d-1)) - \sum_{j=1}^{k-1} (d-1)^{k-1-j} \Psi(E_j(x, d-1)).$$

Also, we know that $E_1(x, d-1) = x$, hence $\Psi(E_1(x, d-1)) = 0$, which gives us

$$\begin{aligned} \mathcal{E}(\mathcal{B}_{d,k}) &= \sum_{j=1}^{k-1} (d-1)^{k-1-j} \Psi(E_{j+1}(x, d-1)) - \sum_{j=1}^{k-1} (d-1)^{k-1-j} \Psi(E_j(x, d-1)) \\ &= \sum_{j=1}^{k-1} (d-1)^{k-1-j} \Big(\Psi(E_{j+1}(x, d-1)) - \Psi(E_j(x, d-1)) \Big). \end{aligned} \tag{33}$$

Taking into consideration Eq. (33), it becomes apparent that in order to prove Eq. (30), it is sufficient to show that

$$\Psi(E_{j+1}(x, d-1)) - \Psi(E_j(x, d-1)) = f_j \sqrt{d-1},$$

for all the $1 \le j \le k-1$. However, this is straightforward to do with the help of Lemma 4. If j is odd, then

$$\begin{aligned} \Psi(E_{j+1}(x, d-1)) - \Psi(E_j(x, d-1)) &= 2\sqrt{d-1} \left(\csc\left(\frac{\pi}{2j+4}\right) - 1 \right) - 2\sqrt{d-1} \left(\cot\left(\frac{\pi}{2j+2}\right) - 1 \right) \\ &= 2\sqrt{d-1} \left(\csc\left(\frac{\pi}{2j+4}\right) - \cot\left(\frac{\pi}{2j+2}\right) \right) = f_j \sqrt{d-1}. \end{aligned}$$

On the other hand, if j is even, we get

$$\begin{aligned} \Psi(E_{j+1}(x, d-1)) - \Psi(E_j(x, d-1)) &= 2\sqrt{d-1} \left(\cot\left(\frac{\pi}{2j+4}\right) - 1 \right) - 2\sqrt{d-1} \left(\csc\left(\frac{\pi}{2j+2}\right) - 1 \right) \\ &= 2\sqrt{d-1} \left(\cot\left(\frac{\pi}{2j+4}\right) - \csc\left(\frac{\pi}{2j+2}\right) \right) = f_j \sqrt{d-1}. \end{aligned}$$

□

Conflict of interest

The author declares that he has no conflict of interest.

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