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Some General Families of Integral Transformations and Related Results

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Abstract. This article is motivated essentially by several extensive developments on the familiar Laplace and Hankel transforms as well as on their extensions and generalizations. Our main object here is to present a number of (presumably new) properties and characteristics as well as inter-relationships among each of such general families of integral transforms as Srivastava's generalized Whittaker transform, Hardy's generalized Hankel transform and Srivastava's ϵ-generalized Hankel transform. Many trivial and inconsequential parametric and argument variations of the classical Laplace transform and its s-multiplied version (or the Laplace-Carson transform), each of which unfortunately is being referred to as a "new" integral transform in the present-day obviously amateurish-type amateurish-type literature, are pointed out. The Srivastava-Panda multidimensional integral transformations involving their multivariable H-function in the kernel as well as the potentially useful process of association of variables in the theory and applications of the multidimensional Laplace transform are also considered with a view to encouraging related further studies and revisits.

1. Introduction, Definitions and Motivation

Named after the French scholar and polymath, Pierre-Simon Laplace (1749–1827), the Laplace transform is defined by

$$\mathcal{L}\{f(t):s\} := \int_0^\infty \mathrm{e}^{-st} f(t)\ \mathrm{d}t =: F_{\mathcal{L}}(s), \tag{1}$$

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provided that the integral exists. Indeed it happens to be one of the most widely-used and extensively-investigated integral transformations. The s-multiplied version of the Laplace transform (or the Laplace-Carson transform):

$$\mathcal{LC}\{f(t):s\} := s\int_0^\infty \mathrm{e}^{-st} f(t)\ \mathrm{d}t =: F_{\mathcal{LC}}(s), \tag{2}$$

which is attributed to the American transmission theorist, John Renshaw Carson (1886–1940), has one distinct advantage over the Laplace transform in the fact that the Laplace-Carson transform of a constant is the same constant itself (see, for details, [14], [17], [27] and [59]).

In the vast literature on the the theory and applications of the Laplace transform (1), one can find a number of its *nontrivial* extensions and generalizations including, for example, those by the Dutch mathematician, Cornelis Simon Meijer (1904–1974) and the Indian mathematician, Rama Shankar Varma (1905–1970). More recently, in the year 1968, by using the Whittaker $W_{\kappa,\mu}$-function defined by (see [15, p. 264, Eq. 6.9 (2)])

$$\begin{aligned} W_{\kappa,\mu}(z) &:= \mathrm{e}^{-\frac{z}{2}}\ z^{\frac{c}{2}}\ \Psi(a,c;z) \\ &= \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\ {}_1F_1\left[\begin{array}{c} a; \\ \\ c; \end{array} z\right] + \frac{\Gamma(c-1)}{\Gamma(a)}\ {}_1F_1\left[\begin{array}{c} a-c+1; \\ \\ 2-c; \end{array} z\right] \\ &\qquad \left(a := \frac{1}{2}-\kappa+\mu;\ c := 2\mu+1\right), \end{aligned} \tag{3}$$

the following generalized Whittaker transform was introduced and studied by Srivastava [38, p. 386, Eq. (1.7)]:

$$\mathcal{S}^{\rho,\sigma}_{q,\kappa,\mu}\{f(t):s\} := \int_0^\infty (st)^{\sigma-\frac{1}{2}}\ \mathrm{e}^{-\frac{1}{2}qst}\ W_{\kappa,\mu}(\rho st)\ f(t)\ \mathrm{d}t =: F_{\mathcal{S}}(s) \tag{4}$$

$$\big(\min\{\Re([q+\rho]s-2\epsilon), \Re(\sigma+\delta'\pm\mu+1)\} > 0\big),$$

where

$$f(t) = \begin{cases} O\big(t^{\delta}\ \mathrm{e}^{\epsilon t}\big) & (t \to 0+) \\ \\ O\big(t^{\delta'}\big) & (t \to \infty). \end{cases}$$

Here, and in what follows, we use the standard notation ${}_{\mathfrak{p}}F_{\mathfrak{q}}$ for a generalized hypergeometric function with $\mathfrak{p}$ numerator parameters and $\mathfrak{q}$ denominator parameters, where

$$\mathfrak{p},\mathfrak{q} \in \mathbb{N}_0 := \mathbb{N}\cup\{0\} \qquad (\mathbb{N} = \{1,2,3,\cdots\}).$$

Indeed, here and in what follows, we use the general Pochhammer symbol or the *shifted factorial* $(\lambda)_\nu$, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined (for $\lambda,\nu \in \mathbb{C}$) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{5}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists. Then, with $\mathfrak{p}$ numerator parameters $\alpha_j \in \mathbb{C}\ \ (j = 1,\cdots,\mathfrak{p})$ and $\mathfrak{q}$ denominator parameters $\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ \ (j = 1,\cdots,\mathfrak{q})$, the generalized hypergeometric function ${}_{\mathfrak{p}}F_{\mathfrak{q}}$ is given by

$$
{}_{\mathfrak{p}}F_{\mathfrak{q}}\left[\begin{array}{c}\alpha_1,\cdots,\alpha_{\mathfrak{p}};\\ \\ \beta_1,\cdots,\beta_{\mathfrak{q}};\end{array} z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_{\mathfrak{p}})_n}{(\beta_1)_n \cdots (\beta_{\mathfrak{q}})_n} \frac{z^n}{n!}
$$
$$
=: {}_{\mathfrak{p}}F_{\mathfrak{q}}\left(\alpha_1,\cdots,\alpha_{\mathfrak{p}};\beta_1,\cdots,\beta_{\mathfrak{q}};z\right), \tag{6}
$$

where, as usual, $\mathbb{C}$ denotes the complex plane and

$$
\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \qquad (\mathbb{Z}^- = \{-1,-2,-3,\cdots\}).
$$

The appropriate conditions for convergence of the infinite series in Eq. (6) are being recalled here as follows (see, for details, [42, p. 3 *et seq.*]):

(i) converges absolutely for $|z| < \infty$ if $\mathfrak{p} \leqq \mathfrak{q}$,
(ii) converges absolutely for $|z| < 1$ if $\mathfrak{p} = \mathfrak{q}+1$, and
(iii) diverges for all z $(z \neq 0)$ if $\mathfrak{p} > \mathfrak{q}+1$.

It is known for the Whittaker $W_{\kappa,\mu}$-function that

$$
W_{\frac{1}{2},\pm\mu} = z^{\mu+\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{2}z}
$$

and

$$
W_{0,\mu} = \sqrt{\frac{2z}{\pi}}\, K_\mu(z),
$$

where, for the modified Bessel (or the Macdonald) function $K_\mu(z)$, we have

$$
K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z}.
$$

Thus, clearly, the generalized Whittaker transform (4) would reduce to the following generalized Laplace transforms:

$$
\mathcal{K}_\nu\{f(t):s\} := \sqrt{\frac{2}{\pi}} \int_0^\infty (st)^{\frac{1}{2}}\, K_\nu(st)\, f(t)\, \mathrm{d}t =: F_{\mathcal{M}K}(s) \tag{7}
$$

and

$$
\mathcal{M}_{\kappa,\mu}\{f(t):s\} := \int_0^\infty (st)^{-\kappa-\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{2}st}\, W_{\kappa+\frac{1}{2},\mu}(st)\, f(t)\, \mathrm{d}t =: F_{\mathcal{M}W}(s), \tag{8}
$$

which were introduced by Meijer (see [28] and [29]). Moreover, the generalized Whittaker transform (4) can easily be seen to reduce also to the following generalizations of the Laplace transform by Varma (see [56] and [57]):

$$
\mathcal{VW}_{\kappa,\mu}\{f(t):s\} := \int_0^\infty (2st)^{-\frac{1}{4}}\, W_{\kappa,\mu}(2st)\, f(t)\, \mathrm{d}t =: F_{\mathcal{V}W}(s) \tag{9}
$$

and

$$
\mathcal{VV}_{\kappa,\mu}\{f(t):s\} := \int_0^\infty (st)^{\mu-\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{2}st}\, W_{\kappa,\mu}(st)\, f(t)\, \mathrm{d}t =: F_{\mathcal{V}V}(s). \tag{10}
$$

It should be remarked in passing that, in its special case when $q = \rho = 1$, Srivastava's generalized Whittaker transform (4) was considered earlier by Mainra [25].

We turn now toward two generalizations of the Hankel transform given by

$$\mathcal{H}_\nu\{f(t):s\} := \int_0^\infty t\, J_\nu(st)\, f(t)\, \mathrm{d}t, \tag{11}$$

where $J_\nu(z)$ denotes the familiar Bessel function defined by (see, for details, [16] and [58])

$$J_\nu(z) := \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}z\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} = \frac{\left(\frac{1}{2}z\right)^{\nu}}{\Gamma(\nu+1)}\, {}_0F_1\left[\begin{matrix} \text{——};\\ \nu+1; \end{matrix}\; -\frac{1}{4}z^2\right]. \tag{12}$$

The first generalization, which is due to the English mathematician, Godfrey Harold Hardy (1877–1947), is defined by (see [19]; see also [16, p. 73])

$$\mathcal{H}_\nu^{(\lambda)}\{f(t):s\} := \int_0^\infty t\, F_\nu(st)\, f(t)\, \mathrm{d}t, \tag{13}$$

where, in terms of the Lommel function $\mathfrak{s}_{\mu,\nu}(z)$, we have

$$\begin{aligned} F_\nu(z) &:= \frac{2^{2-\nu-2\lambda}}{\Gamma(\lambda)\Gamma(\nu+\lambda)}\, \mathfrak{s}_{\nu+2\lambda-1,\nu}(z) \\ &= \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}z\right)^{\nu+2\lambda+2n}}{\Gamma(\lambda+n+1)\Gamma(\nu+\lambda+n+1)} \\ &= \frac{\left(\frac{1}{2}z\right)^{\nu+2\lambda}}{\Gamma(\lambda+1)\Gamma(\nu+\lambda+1)}\, {}_0F_2\left[\begin{matrix} \text{——————};\\ \lambda+1, \nu+\lambda+1; \end{matrix}\; -\frac{1}{4}z^2\right]. \end{aligned} \tag{14}$$

Obviously, in its special case when $\lambda = 0$, Hardy's transform (13) reduces immediately to the Hankel transform (11).

With a view to introducing the aforementioned second generalization of the Hankel transform (11), we recall the function $\psi_{\nu,\lambda,\mu}(z)$ defined, in terms of the Pochhammer symbol in (5), by (see, for details, [35], [36] and [39])

$$\psi_{\nu,\lambda,\mu}(z) := \sqrt{\pi} \sum_{n=0}^\infty \frac{(-1)^n\, (\nu+n+1)_n}{n!\,\Gamma\left(\lambda+n+\frac{1}{2}\right)\Gamma\left(\mu+n+\frac{1}{2}\right)} \left(\frac{1}{4}z\right)^{\frac{1}{2}\nu+n}. \tag{15}$$

By applying the definition (15), the above-mentioned second generalization of the Hankel transform (11) was introduced by Srivastava [36] as follows:

$$\Psi_{\lambda,\mu}^{(\nu)}\{f(t):s\} := \int_0^\infty t\, \psi_{\nu,\lambda+\frac{1}{2},\mu}\left(\frac{1}{4}s^2t^2\right) f(t)\, \mathrm{d}t, \tag{16}$$

where the function $f(t)$ as well as the parameters λ, μ and ν are so constrained that the integral in (16) exists. Clearly, since

$$\psi_{\nu,\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu}\left(z^2\right) = J_\nu(2z),$$

in terms of the Bessel function defined by (12), a special case of the generalized Hankel transform (16) when

$$\lambda = \mu = \frac{1}{2}\nu$$

would yield the Hankel transform defined by (11).

Each of the above-defined integral transforms has been investigated in the literature rather extensively and systematically. For several interesting properties and characteristics of Srivastava's generalized Whittaker transform (4), which was introduced in [38], the reader is referred to the subsequent works by (for example) Srivastava *et al.* (see [37], [39], [40], [41], [45], [50] and [52]; see also [46, p. 289, Eq. 9.4 (53)]), Sinha [34], Munot and Padmanabham [30], Tiwari and Ko [54], Rao [31], Malgonde and Saxena [26], Akhaury [2], and Carmichael and Pathak ([5] and [6]). Motivated essentially by these and other related developments, which are based upon Srivastava's generalized Whittaker transform (4), Hardy's generalized Hankel transform (13) and Srivastava's generalized Hankel transform (16), we propose here to present several (presumably new) properties and characteristics as well as inter-relationships among each of these general families of integral transforms.

Our plan in this paper is summarized as follows. In the next section (Section 2), we present several properties and characteristics, including the inversion theorems and the Parseval-Goldstein type theorems for Srivastava's generalized Whittaker transform (4) as well as for the generalized Hankel transforms (13) and (16). In Section 3, we prove a theorem relating the generalized Hankel transforms defined by (13) and (16). Section 4 establishes a general result which relates Hardy's generalized Hankel transform (13) with Srivastava's generalized Whittaker transform (4). Finally, in the concluding section (Section 5), we first briefly describe our findings in this paper and then point out many trivial and inconsequential parametric and argument variations of the Laplace transform (1) and its aforementioned s-multiplied version (that is, the Laplace-Carson transform), each of which unfortunately is being referred to as a "new" integral transform in the present-day obviously amateurish-type literature. Here, in Section 5 itself, with a view to encouraging and motivating related further studies and revisits, we briefly consider the Srivastava-Panda multidimensional integral transformations involving their multivariable H-function in the kernel as well as the potentially useful process of association of variables in the theory and applications of the multidimensional Laplace transform.

2. Miscellaneous Properties and Characteristics of the General Integral Transforms

We begin this section by presenting the following inversion theorem for Srivastava's generalized Whittaker transform (4).

Theorem 1. (see [38, p. 387, Theorem 1]) *Let*

$$\Phi(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{-\xi}}{\Lambda(1-\xi)}\, \mathrm{d}\xi, \tag{17}$$

where

$$\Lambda(\xi) = \frac{\rho^{\mu+\frac{1}{2}}\, \Gamma(\sigma+\mu+\xi)\Gamma(\sigma-\mu+\xi)}{\left[\frac{1}{2}(q+\rho)\right]^{\sigma+\mu+\xi}\, \Gamma\left(\sigma-\kappa+\xi+\frac{1}{2}\right)} \cdot {}_2F_1\left[\begin{array}{c} \sigma+\mu+\xi, \mu-\kappa+\frac{1}{2}; \\ \sigma-\kappa+\xi+\frac{1}{2}; \end{array} \frac{q-\rho}{q+\rho}\right]. \tag{18}$$

Then the inversion formula for Srivastava's generalized Whittaker transform (4) *is given by*

$$f(t) = \int_0^\infty \Lambda(st)\, \mathcal{S}_{q,\kappa,\mu}^{\rho;\sigma}\{f(t):s\}\, \mathrm{d}s, \tag{19}$$

provided that

$$s^{-c}\, \mathcal{S}_{q,\kappa,\mu}^{\rho;\sigma}\{f(t):s\} \in L(0,\infty) \qquad \text{and} \qquad t^{c-1}\, f(t) \in L(0,R_0) \quad (R_0>0),$$

where $\Re(\sigma \pm \mu + 1) > c > 0$.

We next recall an inversion theorem for Hardy's generalized Hankel transform (13) as Theorem 2 below.

Theorem 2. (see [9]) *In terms of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ of the first and the second kind, let*

$$G_{\nu,\lambda}(z) = \cos(\lambda\pi)\ J_\nu(z) + \sin(\lambda\pi)\ Y_\nu(z). \tag{20}$$

Then the inversion formula for Hardy's generalized Hankel transform (13) *is given by*

$$f(t) = \int_0^\infty s\ G_{\nu,\lambda}(st)\ \mathcal{H}_\nu^{(\lambda)}\{f(t):s\}\ \mathrm{d}s, \tag{21}$$

provided that

(a) $\Re(\lambda+1) > 0,\ \ \Re(\nu+\lambda+1) > 0,\ \ \Re(\nu+2\lambda) < \frac{3}{2},\ \ |\Re(\nu)| \leqq \frac{3}{2};$
(b) $t^\alpha\ f(t) \in L(0, R_1) \quad \left(\alpha = \min\left\{\nu+2\lambda+1, \frac{1}{2}\right\}\ \ (R_1 > 0)\right);$
(c) $t^{\frac{1}{2}}\ f(t) \in L(0, R_2) \quad (R_2 > 0).$

Finally, we state and prove the following inversion theorem for Srivastava's ϵ-generalized Hankel transform (16) in slightly modified form given by

$$\Psi_{\lambda,\mu}^{(\nu,\epsilon)}\{f(t):s\} := \int_0^\infty t\ \mathrm{e}^{-\epsilon st}\ \psi_{\nu,\lambda+\frac{1}{2},\mu}\left(\frac{1}{4}s^2t^2\right)\ f(t)\ \mathrm{d}t, \tag{22}$$

where the function $f(t)$ as well as the parameters λ, μ, ν and ϵ are so constrained that the integral in (22) exists. Obviously, we have

$$\lim_{\epsilon\to 0}\left\{\Psi_{\lambda,\mu}^{(\nu,\epsilon)}\{f(t):s\}\right\} = \Psi_\lambda^{(\nu,\epsilon)}\{f(t):s\}. \tag{23}$$

Theorem 3. *Let*

$$\Theta(x) = \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{c}-\mathrm{i}\infty}^{\mathfrak{c}+\mathrm{i}\infty} \frac{x^{-\xi}}{\Omega(1-\xi)}\ \mathrm{d}\xi, \tag{24}$$

where

$$\Omega(\xi) = \frac{\sqrt{\pi}\ \Gamma(\nu+\xi+1)}{2^{2\nu}\ \epsilon^{\nu+\xi+1}\ \Gamma(\lambda+1)\Gamma\left(\mu+\frac{1}{2}\right)} \cdot {}_3F_3\left[\begin{array}{r} \frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}\nu+1, \frac{1}{2}\nu+\frac{1}{2}\xi+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}\xi+1; \\ \nu+1, \lambda+1, \mu+\frac{1}{2}; \end{array} -\frac{1}{4\epsilon^2}\right]. \tag{25}$$

Then the inversion formula for the ϵ-generalized Hankel transform (22) *is given by*

$$f(t) = \int_0^\infty \Theta(st)\ \Psi_{\lambda,\mu}^{(\nu,\epsilon)}\{f(t):s\}\ \mathrm{d}s, \tag{26}$$

provided that

$$s^{-\mathfrak{c}}\ \Psi_{\lambda,\mu}^{(\nu,\epsilon)}\{f(t):s\} \in L(0,\infty) \qquad \textit{and} \qquad t^{\mathfrak{c}-1}\ f(t) \in L(0,R_3) \quad (R_3 > 0),$$

where $\Re(\epsilon) > 0$ and $\Re(\nu+2) > \mathfrak{c} > 0$.

Proof. Our demonstration of Theorem 3 would run parallel to that of Theorem 1, which is already detailed in [38, p. 388]. Here, in this case, use is made of the de la Vallée Poussin's theorem (see [3, p. 504]) for justifying the order of integration as well as the following known integral formula [17, p. 219, Entry 4.23 (17)]:

$$\int_0^\infty t^{\sigma-1}\, \mathrm{e}^{-st}\ {}_{\mathfrak{p}}F_{\mathfrak{q}}\left[\begin{array}{c}\alpha_1,\cdots,\alpha_{\mathfrak{p}};\\ \\ \beta_1,\cdots,\beta_{\mathfrak{q}};\end{array} \kappa t\right]$$
$$= \frac{\Gamma(\sigma)}{s^\sigma}\ {}_{\mathfrak{p}+1}F_{\mathfrak{q}}\left[\begin{array}{c}\alpha_1,\cdots,\alpha_{\mathfrak{p}},\sigma;\\ \\ \beta_1,\cdots,\beta_{\mathfrak{q}};\end{array} \frac{\kappa}{s}\right], \tag{27}$$

provided that

$$\Re(s) > \begin{cases} 0 & (\mathfrak{p} < \mathfrak{q}) \\ \\ \Re(\kappa) & (\mathfrak{p} = \mathfrak{q}) \end{cases}$$

□

Remark 1. By appropriately specializing the parameters λ, μ and ν, the limit case of Theorem 3 when $|\epsilon| \to 0$ can be simplified considerably to yield an inversin theorem for the generalized Hankel transform (16).

Remark 2. It is fairly straightforward to establish the Parseval-Goldstein type theorem for each of the general families of integral transforms (see [18]; see also [39, p. 318, Theorem 2], [41, p. 266, Theorem 1] and [50, Part I, p. 129, Theorem 3]). The details involved are being left as an exercise for the interested reader.

3. Theorems Relating the Hardy Transform and Srivastava's ϵ-Generalized Hankel Transform

In this section, we state and prove two therems which relate the Hardy transform (13) with the ϵ-generalized Hankel transform (22).

Theorem 4. *Under the hypotheses of Theorem* 3, *let the function* $f(t)$ *be given by*

$$f(t) = \int_0^\infty \Theta(st)\, \Psi_{\zeta,\mu}^{(\delta,\epsilon)}\{f(t):s\}\, \mathrm{d}s, \tag{28}$$

where $\Theta(x)$ *is defined by* (25). *Also let each of the following integrals*:

$$\int_0^{R_4} \left|\eta^{1\pm\nu}\, \mathcal{H}_\nu^{(\lambda)}\{f(t):\eta\}\right| \mathrm{d}\eta \qquad (R_4 > 0)$$

and

$$\int_{R_4}^\infty \left|\eta^{\frac{1}{2}}\, \mathcal{H}_\nu^{(\lambda)}\{f(t):\eta\}\right| \mathrm{d}\eta \qquad (R_4 > 0)$$

be convergent. Then the following relationship holds true between the Hardy transform (13) *with the* ϵ-*generalized Hankel transform* (22):

$$\mathcal{H}_\nu^{(\lambda)}\{g(t)\, f(t):s\} = \int_0^\infty \Psi_{\zeta,\mu}^{(\delta,\epsilon)}\{f(\tau):\eta\}\cdot \mathcal{H}_\nu^{(\lambda)}\{\Theta(\eta t) g(t):s\}\, \mathrm{d}\eta, \tag{29}$$

provided that each member of (29) *exists for a function* $g(t)$ *constrained by*

$$g(t) = \begin{cases} O(t^\chi) & (t \to 0+) \\ \\ O\left(t^\tau\, \mathrm{e}^{\delta t}\right) & (t \to \infty). \end{cases}$$

Proof. Our demonstration of Theorem 4 is based upon the definitions (13) and (28) and would make use of integral inversion which is justified by the de la Vallée Poussin's theorem (see [3, p. 504]) under the hypotheses of Theorem 4. We choose to omit the details involved. □

In a manner analogous to our proofs of Theorem 4 above and Theorem 6 of Section 4, we can establish the following result.

Theorem 5. *Assuming that the hypotheses of Theorem* 2 *are satisfied, let the function* $f(t)$ *be given by*

$$f(t) = \int_0^\infty s\, G_{\nu,\lambda}(st)\, \mathcal{H}_\nu^{(\lambda)}\{f(t) : s\}\, \mathrm{d}s, \tag{30}$$

where $G_{\nu,\lambda}(x)$ *is given by*

$$G_{\nu,\lambda}(z) = \operatorname{cosec}(\nu\pi)\big[\sin\big((\nu+\lambda)\pi\big) J_\nu(z) - \sin(\lambda\pi) J_{-\nu}(z)\big]. \tag{31}$$

Suppose also that each of the following integrals:

$$\int_0^{R_5} \left|\eta^{1\pm\nu}\, \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}\right| \mathrm{d}\eta \qquad (R_5 > 0)$$

and

$$\int_{R_5}^\infty \left|\eta^{\frac{1}{2}}\, \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}\right| \mathrm{d}\eta \qquad (R_5 > 0)$$

is convergent. Then the following relationship holds true between the Hardy transform (13) *and Srivastava's* ϵ*-generalized Hankel transform* (22):

$$\begin{aligned}\Psi_{\zeta,\mu}^{(\delta,\epsilon)}\{g(t)\, f(t) : s\} &= \operatorname{cosec}(\nu\pi) \int_0^\infty \eta\, \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\} \\ &\cdot \big[\sin\big((\nu+\lambda)\pi\big)\, \mathfrak{H}_\nu\big(g(t)\, J_\nu(\eta t); s, \eta\big) - \sin(\lambda\eta)\, \mathfrak{H}_{-\nu}\big(g(t)\, J_{-\nu}(\eta t); s, \eta\big)\big]\, \mathrm{d}\eta,\end{aligned} \tag{32}$$

where

$$\mathfrak{H}_\nu\big(g(t); s, \eta\big) := \Psi_{\zeta,\mu}^{(\delta,\epsilon)}\{g(t)\, J_\nu(\eta t) : s\} \tag{33}$$

and

$$g(t) = \begin{cases} O(t^\chi) & (t \to 0+) \\ O\left(t^\tau\, \mathrm{e}^{\delta t}\right) & (t \to \infty), \end{cases}$$

it being tacitly assumed that each member of the relationship (32) *exists.*

We find it to be worthwhile to remark in passing that both Theorem 4 and Theorem 5 are sufficiently general in nature. Each of these results can indeed be appropriately specialized to deduce a large number of known or new relationships between various simpler integral transforms which we have considered in this paper.

4. Relationship Between the Hardy Transform and Srivastava's Generalized Whittaker Transform

We first state our proposed relationship between the Hardy transform (13) and Srivastava's generalized Whittaker transform (4) as Theorem 6 below.

Theorem 6. *Under the hypotheses of Theorem* 2, *let the function* $f(t)$ *be given by*

$$f(t) = \int_0^\infty s\ G_{\nu,\lambda}(st)\ \mathcal{H}_\nu^{(\lambda)}\{f(t) : s\}\ \mathrm{d}s. \tag{34}$$

Suppose also that each of the following integrals:

$$\int_0^{R_6} \left|\eta^{1\pm\nu}\ \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}\right|\ \mathrm{d}\eta \qquad (R_6 > 0)$$

and

$$\int_{R_6}^\infty \left|\eta^{\frac{1}{2}}\ \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}\right|\ \mathrm{d}\eta \qquad (R_6 > 0)$$

is convergent. If

$$\Re\big((\rho+q)s\big) > 0 \qquad \Re(\chi+\sigma+1) > |\Re(\nu) + |\Re(\mu)|,$$

then the following relationship holds true:

$$\mathcal{S}_{q,\kappa,\mu}^{(\rho,\sigma)}\{g(t)\ f(t) : s\} = \operatorname{cosec}(\nu\pi) \int_0^\infty \eta\ \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}$$
$$\cdot \left[\sin\big((\nu+\lambda)\pi\big)\ \mathfrak{h}_\nu\big(g(t)\ J_\nu(\eta t); s, \eta\big) - \sin(\lambda\eta)\ \mathfrak{h}_{-\nu}\big(g(t)\ J_{-\nu}(\eta t); s, \eta\big)\right] \mathrm{d}\eta, \tag{35}$$

where

$$\mathfrak{h}_\nu\big(g(t); s, \eta\big) := \mathcal{S}_{q,\kappa,\mu}^{(\rho;\sigma)}\{g(t)\ J_\nu(\eta t) : s\} \tag{36}$$

and

$$g(t) = \begin{cases} O(t^\chi) & (t \to 0+) \\ O\left(t^\tau\ \mathrm{e}^{\delta t}\right) & (t \to \infty). \end{cases}$$

Proof. First of all, since

$$Y_\nu(z) = \operatorname{cosec}(\nu\pi)\big[J_\nu(z)\ \cos(\nu\pi) - J_{-\nu}(z),$$

it is easily seen from (20) that

$$G_{\nu,\lambda}(z) = \operatorname{cosec}(\nu\pi)\left[\sin\big((\nu+\lambda)\pi\big)J_\nu(z) - \sin(\lambda\pi)J_{-\nu}(z)\right]. \tag{37}$$

Upon substituting for $f(t)$ from (34), if we make use of the formula (37) and invert the order of integration in the resulting double integrals, we obtain

$$\sin(\nu\pi)\ \mathcal{S}_{q,\kappa,\mu}^{(\rho,\sigma)}\{g(t)\ f(t) : s\} = \sin\big((\nu+\lambda)\pi\big)$$
$$\cdot \int_0^\infty \eta\ \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}\ \mathcal{S}_{q,\kappa,\mu}^{(\rho;\sigma)}\{g(t)\ J_\nu(\eta t) : s\}\ \mathrm{d}\eta$$
$$- \sin(\lambda\pi) \int_0^\infty \eta\ \mathcal{H}_\nu^{(\lambda)}\{f(t) : \eta\}\ \mathcal{S}_{q,\kappa,\mu}^{(\rho;\sigma)}\{g(t)\ J_{-\nu}(\eta t) : s\}\ \mathrm{d}\eta. \tag{38}$$

The above-mentioned inversion of the order of integration is justifiable by appealing to the de la Vallée Poussin's theorem (see [3, p. 504]) under the hypotheses of Theorem 6. The final result (35) would now follow from (38) in light of the definition of $\mathfrak{h}_\nu\big(g(t); s, \eta\big)$ in (36). □

Remark 3. Since (see [38, p. 387, Eq. (1.13)])

$$\begin{aligned}\mathfrak{h}_\nu\big(t^\chi;s,\eta\big) &:= \mathcal{S}^{(\rho,\sigma)}_{q,\kappa,\mu}\{t^\chi\ J_\nu(\eta t):s\} = \frac{s^{\sigma+\mu}\ \rho^{\mu+\frac{1}{2}}\ \left(\frac{1}{2}\eta\right)^\nu}{\left[\frac{1}{2}(\rho+q)s\right]^{\chi+\nu+\sigma+\mu+1}} \\ &\cdot \sum_{n=0}^{\infty} \frac{(-1)^n\ \Gamma(\chi+\nu+\sigma\pm\mu+2n+1)}{n!\ \Gamma\left(\chi+\nu+\sigma-\kappa+2n+\frac{3}{2}\right)\Gamma(\nu+n+1)} \left(\frac{\eta}{(\rho+q)s}\right)^{2n} \\ &\cdot\ {}_2F_1\left[\begin{array}{r} \chi+\nu+\sigma+\mu+2n+1,\mu-\kappa+\frac{1}{2};\\ \chi+\nu+\sigma-\kappa+2n+\frac{3}{2}; \end{array} \frac{q-\rho}{q+\rho}\right], \end{aligned} \tag{39}$$

which holds true when

$$\Re\big((\rho+q)s\big)>0 \qquad \text{and} \qquad \Re(\chi+\nu+\sigma\pm\mu+1)>0,$$

in its special case when $g(t)=t^\chi$, Theorem 6 would correspond to a known result [38, p. 389, Theorem 2].

5. Further Remarks and Observations

Our investigation in this article is motivated essentially by a number of extensive developments on the familiar Laplace and Hankel transforms as well as on the extensions and generalizations of each of these integral transforms. Here, in this article, we have presented several (presumably new) properties and characteristics as well as inter-relationships among each of such general families of integral transforms as Srivastava's generalized Whittaker transform, Hardy's generalized Hankel transform and Srivastava's ϵ-generalized Hankel transform. Some of our main results (especially Theorem 4, Theorem 5 and Theorem 6) have been stated and proved here in a sufficiently general form. Each of these three results can indeed be appropriately specialized to deduce a large number of known or new relationships between various simpler integral transforms which we have considered in this paper.

We now turn to many trivial and inconsequential parametric and argument variations of the classical Laplace transform (1) and its s-multiplied version (2). Unfortunately, all of these numerous trivial and inconsequential parametric and argument variations of the classical Laplace transform (1) and its s-multiplied version (2) are being claimed, by a number of obviously amateurish-type authors, to be a "new" integral transform in the present-day literature. Some of these trivial and inconsequential parametric and argument variations of the classical Laplace transform (1) and its s-multiplied version (2) are being listed below (see also [43]):

I. The "Sumudu" Transform. The so-called *Sumudu* transform is an integral transform defined by

$$G(u)=\mathfrak{S}\{f(t);u\}:=\int_0^\infty \mathrm{e}^{-t}\ f(ut)\ \mathrm{d}t \qquad (-\tau_1<u<\tau_2), \tag{40}$$

which, when compared with the definitions in (1) and (2), leads us to the following straightforward relationships with the Laplace transform and the Laplace-Carson transform:

$$G\left(\frac{1}{s}\right)=s\ F_{\mathcal{L}}(s) \qquad \text{or} \qquad G(u)=\frac{1}{u}\ F_{\mathcal{L}}\left(\frac{1}{u}\right). \tag{41}$$

and

$$G\left(\frac{1}{s}\right)=F_{\mathcal{LC}}(s) \qquad \text{or} \qquad G(u)=F_{\mathcal{LC}}\left(\frac{1}{u}\right). \tag{42}$$

II. The "Natural" (or $\mathcal{N}$-) Transform. The so-called *natural* (or $\mathcal{N}$-) transform is defined by (see [20])

$$R(u,s)=\mathcal{N}\{f(t):u,s\}:=\int_0^\infty \mathrm{e}^{-st}\ f(ut)\ \mathrm{d}t, \tag{43}$$

which obviously reduces to the Laplace transform in (1) when $u = 1$ and the Sumudu transform in (40) when $s = 1$. However, by the following rather trivial change of variable in (43):

$$t = \frac{\tau}{u} \qquad \text{and} \qquad \mathrm{d}t = \frac{\mathrm{d}\tau}{u} \qquad \big(\Re(u) > 0\big),$$

one can easily see that

$$\begin{aligned}\mathcal{N}\{f(t) : u, s\} &= \frac{1}{u} \int_0^\infty \mathrm{e}^{-\frac{s\tau}{u}} \, f(\tau) \, \mathrm{d}\tau \\ &= \frac{1}{u} \mathcal{L}\left\{ f(\tau) : \frac{s}{u} \right\},\end{aligned} \tag{44}$$

which provides a *direct* (non-specialized) relationship with the classical Laplace transform in (1). Much more trivially, we have

$$\mathcal{N}\{f(t) : u, s\} = \mathcal{L}\left\{ f\left(ut\right) : s \right\} \qquad \big(\min\{\Re(s), \Re(u)\} > 0\big). \tag{45}$$

Clearly, the equations (44) and (45) exhibit the fact that each and every usage of the classical Laplace transform can be translated rather trivially in terms of the so-called *natural* (or $\mathcal{N}$-) transform defined by (43). Some of the recent usages of the *natural* (or $\mathcal{N}$-) transform or its further inconsequential k-variation can be found in [32], [33] and [55]. In particular, Sene and Srivastava [32] made use of the following k-version of the *natural* (or $\mathcal{N}$-) transform defined by (43):

$$\mathcal{L}_k\{f(t) : s\} := \int_0^\infty \mathrm{e}^{-\frac{st^k}{k}} \, f(t) \, \frac{\mathrm{d}t}{t^{1-k}}, \tag{46}$$

which, under the change of the variable t as follows:

$$t = \tau^{\frac{1}{k}} \qquad \text{and} \qquad \mathrm{d}t = \frac{1}{k} \, \tau^{\frac{1}{k}-1} \, \mathrm{d}\tau \qquad (k > 0),$$

yields

$$\begin{aligned}\mathcal{L}_k\{f(t) : s\} &= \frac{1}{k} \int_0^\infty \mathrm{e}^{-\frac{s\,\tau}{k}} \, f\left(\tau^{\frac{1}{k}}\right) \, \mathrm{d}\tau \\ &= \frac{1}{k} \, \mathcal{L}\left\{ f\left(t^{\frac{1}{k}}\right) : \frac{s}{k} \right\} \qquad (k > 0)\end{aligned} \tag{47}$$

in terms of the classical Laplace transform (1) itself. On the other hand, Valizadeh *et al.* [55] and Shah *et al.* [33] used, respectively, the versions (43) and (44) of the so-called *natural* transform.

III. The "Pathway" (or $\mathcal{P}_\delta$-Transform. The so-called *pathway* (or $\mathcal{P}_\delta$-) transform is defined, for a function $f(t)$ $(t \in \mathbb{R})$, by (see [23]; see also [44] and [53])

$$\mathcal{P}_\delta\{f(t) : s\} = F_{\mathcal{P}}(s) := \int_0^\infty [1 + (\delta - 1)s]^{-\frac{t}{\delta-1}} \, f(t)\mathrm{d}t \qquad (\delta > 1), \tag{48}$$

provided that the sufficient existence conditions are satisfied.)

Indeed, upon closely comparing the definitions in (48) and (1), it is easily observed that the so-called $\mathcal{P}_\delta$-transform is essentially the same as the classical Laplace transform *with* the following rather trivial parameter change in (1):

$$s \longmapsto \frac{\ln[1 + (\delta - 1)s]}{\delta - 1} \qquad (\delta > 1). \tag{49}$$

In fact, the following relationship holds true between the so-called $\mathcal{P}_\delta$-transform defined by (48) and the classical Laplace transform given by (1):

$$\mathcal{P}_\delta \{f(t) : s\} = \mathfrak{L}\left\{f(t) : \left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)\right\} \qquad (\delta > 1) \tag{50}$$

or, equivalently, by

$$\mathfrak{L}\{f(t) : s\} = \mathcal{P}_\delta \left\{f(t) : \frac{e^{(\delta-1)s}-1}{\delta-1}\right\} \qquad (\delta > 1), \tag{51}$$

which can indeed be applied reasonably simply to convert the table of the Laplace transforms into the corresponding table of the $\mathcal{P}_\delta$-transform and *vice versa.* However, in spite of the *easy-to-use* relationships (50) and (51), the current literature on integral transforms, special functions and fractional calculus is flooded by investigations claiming at least implicitly that the $\mathcal{P}_\delta$-transform $\mathcal{P}_\delta\{f(t) : s\}$ defined by (48) is a generalization of the classical Laplace transform defined by (1) (see, for example, [1]).

Other examples of several rather trivial and inconsequential parameter and argument variations of the classical Laplace transform (with the kernel e^{-st}) include the so-called *Sadik* transform (with the kernel $\frac{1}{v^\beta}\,\mathrm{e}^{-v^\alpha\,t}$), which, for $\beta = \alpha$, is simply the $\frac{1}{s}$-multiplied Laplace transform when we replace the parameter s trivially by v^α. Thus, clearly, we have

$$\begin{aligned}\mathbb{L}_{v^\beta}\{f(t) : v^\alpha\} &:= \frac{1}{v^\beta}\int_0^\infty \mathrm{e}^{-v^\alpha\,t}\, f(t)\,\mathrm{d}t \\ &= \frac{1}{v^\beta}\,\mathcal{L}\{f(t) : v^\alpha\}\end{aligned} \tag{52}$$

in terms of the classical Laplace transform with $s = v^\alpha$ and multiplied trivially by $\frac{1}{v^\beta}$. In what sense, if at all, does (52) define a nontrivial and obviously inconsequential generalization of the classical Laplace transform?

Such other rather trivial and inconsequential parameter and argument variations of the classical Laplace transform in (1) or the Laplace-Carson transform (2) are known as the above-mentioned *Sumudu* transform (when $\alpha = -1$ and $\beta = 1$), the so-called *Elzaki* transform (when $\beta = \alpha = -1$), the so-called *Tang* transform (when $\alpha = -2$ and $\beta = 1$), the so-called *Kamal* transform (when $\alpha + 1 = \beta = 0$), the so-called *Aboodh* transform (when $\alpha = \beta = 1$), and so on. So far there are no convincing arguments as to why all these rather trivial and inconsequential parameter and argument variations of the classical Laplace transform (1) or the Laplace-Carson transform (2) are preferable in any way to the the classical Laplace transform in (1) or the Laplace-Carson transform (2) themselves. Such unsubstantiated claims and attempts to simply translate and repeat known theories and known applications of the classical Laplace transform in terms of the above-mentioned (and possibly many other) obviously trivial and inconsequential parameter and argument variations of the classical Laplace transform (1) or the Laplace-Carson transform (2) ought to be discouraged by all means.

Finally, we choose to reiterate the fact that the need for *simultaneous* operational calculus (based upon multidimensional integral transformations) presents itself quite naturally when problems dependent on several variables are to be treated operationally (see, for example, [4], [12] and [13]; see also [10]). Moreover, since a wide variety of mathematical functions, which occur rather frequently in problems in applied mathematics and mathematical analysis, are special cases of the Srivastava-Panda H-function of several complex variables (see, for details, [48] and [49]), a systematic further study of the Srivastava-Panda multidimensional integral transformations involving their multivariable H-function in the kernel (see [50] and [51]) is believed to lead to deeper, general and potentially useful results. We recall here a very specialized case, that is, the multidimensional Laplace transform defined by

$$\begin{aligned}
&\mathcal{L}_n\{f(t_1,\cdots,t_n):s_1,\cdots s_n\} \\
&\qquad :=\int_0^\infty\cdots\int_0^\infty \exp\left(-s_1t_1-\cdots-s_nt_n\right)\; f(t_1,\cdots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&\qquad =: F_{\mathcal{L}_n}\left(s_1,\cdots,s_n\right),
\end{aligned} \tag{53}$$

which possesses the following inversion formula:

$$\begin{aligned}
f(t_1,\cdots,t_n)=\mathcal{L}_n^{-1}\{F_{\mathcal{L}_n}\left(s_1,\cdots,s_n\right)\}=\frac{1}{(2\pi\mathrm{i})^n}\int\limits_{\sigma_1-\mathrm{i}\infty}^{\sigma_1+\mathrm{i}\infty}\cdots\int\limits_{\sigma_n-\mathrm{i}\infty}^{\sigma_n+\mathrm{i}\infty}\exp\left(s_1t_1+\cdots+s_nt_n\right) \\
\cdot F_{\mathcal{L}_n}\left(s_1,\cdots,s_n\right)\,\mathrm{d}s_1\cdots\mathrm{d}s_n \qquad (\min\{\sigma_1,\cdots,\sigma_n\}>0).
\end{aligned} \tag{54}$$

Indeed, in some types of Systems Analysis, one needs to find the n-dimensional inverse Laplace transform of the function $F_{\mathcal{L}_n}\left(s_1,\cdots,s_n\right)$, as given by (54), and then to evaluate it at $t_1=\cdots=t_n=t$, that is, to find the function $\mathfrak{g}(t)$ given by

$$\mathfrak{g}(t):=f(t,\cdots,t)=\mathcal{L}_n^{-1}\{F_{\mathcal{L}_n}\left(s_1,\cdots,s_n\right)\}\big|_{t_1=\cdots=t_n=t}. \tag{55}$$

Thus, if

$$\mathfrak{G}(s)=\mathcal{L}\{\mathfrak{g}(t):s\}:=\int_0^\infty \mathrm{e}^{-st}\,\mathfrak{g}(t)\,\mathrm{d}t, \tag{56}$$

we say that $\mathfrak{G}(s)$ is the *associated transform* of $F_{\mathcal{L}_n}\left(s_1,\cdots,s_n\right)$ (see, for details, [24]). Such processes of *association of variables* in the theory and applications of the multidimensional Laplace transform (53) have been investigated extensively (see, for example, [7], [8], [11], [21] and [22]) and seem to deserve further studies and revisits.

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