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On the core inverse

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Abstract. We consider the core inverse of a bounded Hilbert space operator, and extend some results from (H. Wang, X. Liu, Linear Multilin. Algebra 63 (2015)) to infinite dimensional settings.

1. Introduction

We consider arbitrary Hilbert spaces (say H, K). Let $\mathcal{L}(H, K)$ denote the set of all bounded linear operators from H to K, and abbreviate $\mathcal{L}(H) = \mathcal{L}(H, H)$. If $A \in \mathcal{L}(H, K)$, then $\mathcal{R}(A)$ is the range, and $\mathcal{N}(A)$ is the null-space of A. We use $\dim_h(H)$ to denote the orthogonal dimension of a Hilbert space H. The same notation is allowed for closed subspaces of a Hilbert space.

If $A \in \mathcal{L}(H, K)$ and $\mathcal{R}(A)$ is closed, then we define the rank of A as follows: $\mathrm{rk}(A) = \dim_h(\mathcal{R}(A)$. This obviously extends the standard definition of a rank of a matrix to infinite dimensional settings.

For $A \in \mathcal{L}(H)$, the ascent of A, denoted by $\mathrm{asc}(A)$, is the smallest non-negative integer k (if it exists) satisfying $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$. The descent of A, denoted by $\mathrm{dsc}(A)$, is the smallest non-negative integer k satisfying $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. If $\mathrm{asc}(A) < \infty$ and $\mathrm{dsc}(A) < \infty$, then $\mathrm{asc}(A) = \mathrm{dsc}(A)$ and this common value is the Drazin index of A, denoted by $\mathrm{ind}(A)$. Notice that A is invertible if and only if $\mathrm{ind}(A) = 0$. More generally, $\mathrm{ind}(A) \leq 1$ if and only if A is group invertible, i.e. if and only if there exists (necessarily unique) the group inverse $A^\# \in \mathcal{L}(H)$ of A, satisfying

$$AA^\#A = A, \quad A^\#AA^\# = A^\#, \quad AA^\# = A^\#A.$$

If A is group invertible, then $\mathcal{R}(A)$ is closed, and $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$, knowing that this direct sum is not necessarily orthogonal.

Let $A \in \mathcal{L}(H)$. Following [1], [3], and [4], the operator $A^{\textcircled{\#}}$ is the core inverse of A, if:

$$AA^{\textcircled{\#}}A = A, \quad \mathcal{R}(A^{\textcircled{\#}}) = \mathcal{R}(A), \quad \mathcal{N}(A^{\textcircled{\#}}) = \mathcal{N}(A).$$

It immediately follows that if A is core invertible, then $\mathcal{R}(A)$ is closed.

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Let $A \in \mathcal{L}(H, K)$. The Moore-Penrose inverse of A is the unique operator (in the case when it exists) $A^\dagger \in \mathcal{L}(K, H)$ satisfying:

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

$A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed in K.

It is well-known that if $A^{\textcircled{\#}}$ exists for $A \in \mathcal{L}(H)$, then $A^{\textcircled{\#}} = A^\# AA^\dagger$ [1].

Finally if $A \in \mathcal{L}(H, K)$, then $X \in \mathcal{L}(K, H)$ has the property $X \in A\{1, 3\}$ if and only if $AXA = A$ and $(AX)^* = AX$.

2. Results

We extend the result [2](Lemma 3.1) to infinite dimensional settings.

Theorem 2.1. *Let H_1, H_2, H_3 be Hilbert spaces, and let $A \in \mathcal{L}(H_1, H_3)$, $B \in \mathcal{L}(H_2, H_3)$. Suppose that operators A, B have closed ranges. Consider operators:*

$$M = \begin{bmatrix} A & B \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \to H_3 \qquad \text{and} \qquad N = (I - AA^\dagger)B.$$

If M, N have closed ranges, then

$$\mathrm{rk}(M) = \mathrm{rk}(N) + \mathrm{rk}(A).$$

Proof. Let $H_1 = \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$, $H_2 = \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}$ and $H_3 = \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$. Then we have the following matrix decompositions of operators:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $A_1 \in \mathcal{L}(\mathcal{R}(A^*), \mathcal{R}(A))$ is invertible, and

$$B = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A^*) \end{bmatrix}.$$

We see that

$$\mathcal{R}(M) = \left\{ \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} : x \in H_1, y \in H_2 \right\} = \mathcal{R}(A) + \mathcal{R}(B).$$

On the other hand,

$$N = \begin{bmatrix} 0 & 0 \\ B_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

implying that

$$\mathcal{R}(N) = \mathcal{R}(B_2) \subset \mathcal{N}(A^*).$$

Hence, $\mathrm{rk}(N) = \mathrm{rk}(B_2)$. Since $\mathcal{R}(B_2) \subset \mathcal{N}(A^*)$, we get

$$\mathrm{rk}(M) = \dim(\mathcal{R}(A) + \mathcal{R}(B)) = \dim \mathcal{R}(A) + \dim \mathcal{R}(B_2) = \mathrm{rk}(A) + \mathrm{rk}(N).$$

The second equality follows form the following consideration:

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) + \mathcal{R}(B_1 + B_2) \subset \mathcal{R}(A) + \mathcal{R}(B_1) + \mathcal{R}(B_2) = \mathcal{R}(A) + \mathcal{R}(B_2).$$

Suppose that $x \in \mathcal{R}(A) + \mathcal{R}(B_2)$. Then $x = A_1 y + B_2 z$ with $y \in \mathcal{R}(A^*)$ and $z \in \mathcal{R}(B^*)$. Then we have

$$x = A_1 y - B_1 z + B_1 z + B_2 z = A_1(y - A_1^{-1} B_1 z) + Bz \in \mathcal{R}(A) + \mathcal{R}(B).$$

Thus, we have proved

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) + \mathcal{R}(B_2).$$

□

Now, the results from [2] (Theorem 3.2) are valid in infinite dimensional settings. We also use some results from [3]. Notice that the equality (1) in the following theorem is actually the reverse order law for the core inverse.

Theorem 2.2. *Let $A, B, AB \in \mathcal{L}(H)$. If A, B, AB are core invertible, then the following statements are equivalent:*

(1) $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$;

(2) $B^*A^*A\left(I - BB^{\textcircled{\#}}\right)\ A = 0$ *and* $R\left(B^{\textcircled{\#}}A\right) \subseteq \mathcal{R}(AB)$.

Proof. We have two approaches. The first is to use Theorem 2.1 and then to adjust the proof from [2] (Theorem 3.2) to ranks of arbitrary cardinalities. To avoid repetition, we prove the result in the second way.

(1)⇒(2): From $A^{\textcircled{\#}} = A^{\sharp}AA^{\dagger}$ and (1) we get

$$B^*A^*ABB^{\textcircled{\#}}A^{\textcircled{\#}} = B^*A^* \quad \text{and} \quad AB(AB)^{\dagger}B^{\textcircled{\#}}A^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}. \tag{2.1}$$

We multiply the first equation in (2.1) by A^2 from the right side, and obtain

$$B^*A^*ABB^{\textcircled{\#}}A^{\textcircled{\#}}A^2 = B^*A^*A^2$$

Since $A^{\textcircled{\#}}A^2 = A$, we get

$$B^*A^*ABB^{\textcircled{\#}}A = B^*A^*A^2,$$

which is equivalent to

$$B^*A^*A\left(I - BB^{\textcircled{\#}}\right)A = 0.$$

We multiply the second equality in (2.1) by A^2 from the right side and obtain

$$(I - AB(AB)^{\dagger})B^{\textcircled{\#}}A = 0,$$

wherefrom

$$\mathcal{R}(B^{\textcircled{\#}}A) \subseteq \mathcal{N}(I - AB(AB)^{\dagger}) = \mathcal{R}(AB(AB)^{\dagger}) = \mathcal{R}(AB).$$

(2)⇒(1): Since (2) holds, we get

$$B^*A^*ABB^{\textcircled{\#}}A = B^*A^*A^2,$$

We multiply by $A^{\sharp}A^{\dagger}$ the last equation from the right side, and get

$$B^*A^*ABB^{\textcircled{\#}}AA^{\sharp}A^{\dagger} = B^*A^*A^2A^{\sharp}A^{\dagger},$$

so

$$B^*A^*ABB^{\textcircled{\#}}A^{\textcircled{\#}} = B^*A^*AA^{\dagger} = B^*A^*.$$

We multiply the last equality by $((AB)^*)^{\dagger}$ from the left side and obtain

$$ABB^{\textcircled{\#}}A^{\textcircled{\#}} = AB(AB)^{\dagger},$$

and consequently

$$(AB)^{\sharp}ABB^{\textcircled{\#}}A^{\textcircled{\#}} = (AB)^{\sharp}AB(AB)^{\dagger}. \tag{2.2}$$

We have the following:

$$\mathcal{R}(B^{\textcircled{\#}}A^{\textcircled{\#}}) = \mathcal{R}(B^{\textcircled{\#}}A^{\textcircled{\#}}) = \mathcal{R}(B^{\textcircled{\#}}AA^{\textcircled{\#}}A^{\dagger}) \subset \mathcal{R}(B^{\textcircled{\#}}A) \subset \mathcal{R}(AB). \tag{2.3}$$

Notice that

$$[AB(AB)^{\sharp}]_{\mathcal{R}(AB)} = I|_{\mathcal{R}(AB)}.$$

Thus, from (2.2) and (2.3) we have

$$(AB)^{\textcircled{\#}} = (AB)^{\sharp}AB(AB)^{\dagger} = B^{\textcircled{\#}}A^{\textcircled{\#}}.$$

In this way the proof is completed. □

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